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Invariants of Differential Geometry by the Use of Vector Forms.

By C. D. Rice.

I. *Introduction.*

Let the equation of the surface be given by

$$x=f(uv), \tag{1}$$

where u and v are scalar variables. Partial derivatives in what follows with respect to u and v are represented by the subscripts 1 and 2 respectively. Derivatives with respect to s, the length of a curve on the surface, will be denoted by primes. The well-known constants in a point are given by

$$\left.\begin{array}{lll} E=-Sx_1x_1, & F=-Sx_1x_2, & G=-Sx_2x_2, \\ L=Sa_1x_1=-Sax_{11}, & M=Sa_1x_2=Sa_2x_1=-Sax_{12}, & N=Sa_2x_2=-Sax_{22}, \end{array}\right\} \tag{2}$$

where a is the unit vector normal to the surface at the point x. The vectors x', x_1, x_2, a', a_1, a_2 are all parallel to the tangent plane at the point. When a curve upon the surface is determined by a scalar relation

$$\phi(uv)=c,$$

we have

$$\phi_1u'+\phi_2v'=0,$$

or

$$\frac{\phi_2}{u'}=-\frac{\phi_1}{v'}=r. \tag{3}$$

By the use of these relations we find from

$$x'=x_1u'+x_2v'$$

the relation

$$rx'=\phi_2x_1-\phi_1x_2=w. \tag{4}$$

In like manner we find

$$ra'=\phi_2a_1-\phi_1a_2=\overline{\omega}. \tag{5}$$

From the study of surfaces we have

$$\Delta^2=EG-F^2=Sx_1x_1Sx_2x_2-Sx_1x_2Sx_1x_2=-SVx_1x_2Vx_1x_2. \tag{6}$$

But we know that $\lambda a = Vx_1x_2$, and hence we find

$$\Delta a = Vx_1x_2. \tag{7}$$

By multiplication we find also

$$\Delta = -Sax_1x_2. \tag{8}$$

In like manner we write $\Delta_0 a = Va_1a_2$ and find

$$\Delta_0 = -Saa_1a_2. \tag{9}$$

But we know that the Gaussian curvature k is given by

$$\begin{aligned}\Delta k &= \frac{1}{\Delta}(LN - M^2)\\ &= \frac{1}{\Delta}(Sa_1x_1Sa_2x_2 - Sa_1x_2Sa_2x_1), \text{ since } Sa_2x_1 = Sa_1x_2 = M,\\ &= -\frac{1}{\Delta}SVa_1a_2Vx_1x_2 = -Saa_1a_2. \end{aligned}\tag{10}$$

$$\therefore \Delta k = \Delta_0. \tag{11}$$

By the use of r and w defined above we have the quadratic form:

$$\begin{aligned}\theta &= E\phi_2^2 - 2F\phi_2\phi_1 + G\phi_1^2 = -S(\phi_2x_1 - \phi_1x_2)(\phi_2x_1 - \phi_1x_2) = -Sww\\ &= -r^2Sx'x', \text{ since } w = \phi_2x_1 - \phi_1x_2 = rx',\\ &= r^2, \text{ since } Sx'x' = -1.\end{aligned}\tag{12}$$

Also we have the quadratic:

$$\begin{aligned}\chi &= L\phi_2^2 - 2M\phi_2\phi_1 + N\phi_1^2 = S(\phi_2x_1 - \phi_1x_2)(\phi_2a_1 - \phi_1a_2) = Sw\bar{\omega}\\ &= r^2Sa'x', \text{ since } ra' = \bar{\omega} \text{ and } rx' = w,\\ &= r^2\frac{1}{\rho},\end{aligned}\tag{13}$$

where $\frac{1}{\rho} = Sa'x'$ is the curvature of a normal section through the tangent at the point.

Many expressions may be abbreviated by the use of the operator $\frac{d}{dn}$ where dn is an element of the arc of the curve on the surface at right angles to the given curve.

At any point P of the curve on the surface we have x', the unit vector along the tangent to the curve, and a the unit vector normal to the surface. Let us take ξ' a unit vector in the tangent plane at right angles to x. Then we have

$$\xi' = \frac{dx}{dn} = Vax'.$$

Let us write $\xi'=cx_1+ex_2$ where c and e are scalars to be determined. We find

$$Sax_1\xi'=eSax_1x_2, \qquad Sax_2\xi'=cSax_2x_1,$$
$$\therefore\ \Delta e=-Sax_1\xi'=-Sax_1Vax'=-Sx_1x', \qquad \therefore\ \Delta c=Sax_2\xi'=Sax_2Vax'=Sx_2x'.$$

Hence we have

$$\xi'=-\frac{1}{\Delta}\{x_2Sx_1x'-x_1Sx_2x'\},\ \text{or}\ \frac{dx}{dn}=-\frac{1}{\Delta}\left\{\frac{\partial x}{\partial v}Sx_1x'-\frac{\partial x}{\partial u}Sx_2x'\right\}.$$

If now R be any function of u and v we find

$$\frac{dR}{dn}=\frac{dR}{dx}\cdot\frac{dx}{dn}=-\frac{1}{\Delta}\left\{\frac{\partial R}{\partial v}Sx_1x'-\frac{\partial R}{\partial u}Sx_2x'\right\}. \tag{14}$$

The operator $\frac{d}{dn}$ is very useful in what follows. In particular we have

$$\Delta\frac{d\phi}{dn}=-\{\phi_2Sx'x_1-\phi_1Sx'x_2\}=-\{Sx'(\phi_2x_1-\phi_1x_2)\}=-Sx'w=r,$$

$$\therefore\ \frac{d\phi}{dn}=\frac{r}{\Delta}\equiv\beta, \tag{15}$$

where we define the quantity β by $\beta=\frac{r}{\Delta}$.

From a study of curves on surfaces we find

$$D=-Sax'x'' \tag{16}$$

to be the *Geodesic curvature*, and the expression

$$W=-Saa'x' \tag{17}$$

to be the *Geodesic torsion.*

Also we have the cubic

$$K=Sa''x'-Sa'x''=Pu'^3+3Qu'^2v'+3Ru'v'^2+Sv'^3, \tag{18}$$

where

$$\left.\begin{aligned} P&=Sa_{11}x_1-Sx_{11}a_1,\quad Q=Sa_{12}x_1-Sx_{12}a_1=Sa_{11}x_2-Sx_{11}a_2,\\ R&=Sa_{12}x_2-Sx_{12}a_2=Sa_{22}x_1-Sx_{22}a_1,\quad S=Sa_{22}x_2-Sx_{22}a_2.\end{aligned}\right\} \tag{19}$$

The mean curvature h is given by

$$\begin{aligned}\Delta h&=\frac{1}{\Delta}\{EN-2FM+GL\}\\ &=-\frac{1}{\Delta}\{(Sx_1x_1Sa_2x_2-Sx_1x_2Sa_2x_1)-(Sx_1x_2Sa_1x_2-Sx_2x_2Sa_1x_1)\}\\ &=\frac{1}{\Delta}\{SVx_1x_2Va_1x_2-SVx_1x_2Va_2x_1\}=Saa_1x_2-Saa_2x_1.\end{aligned} \tag{20}$$

From this expression for h we easily find

$$\left.\begin{aligned} \Delta^2 h_1 &= ER-2FQ+GP, \quad \Delta^2 h_2 = ES-2FR+GQ, \\ \Delta^2 h' &= E(Ru'+Sv')-2F(Qu'+Rv')+G(Pu'+Qv'). \end{aligned}\right\} \tag{21}$$

Change of Parameters.—Suppose we have the surface $\bar{x}=f(\bar{u}\bar{v})$ and wish to change to the parameters u, v, where $u=P(\bar{u}\bar{v})$, $v=Q(\bar{u}\bar{v})$. Let us write,

$$\frac{\partial u}{\partial \bar{u}}=\frac{\partial P}{\partial \bar{u}}=P_1, \quad \frac{\partial u}{\partial \bar{v}}=\frac{\partial P}{\partial \bar{v}}=P_2, \quad \frac{\partial v}{\partial \bar{u}}=\frac{\partial Q}{\partial \bar{u}}=Q_1, \quad \frac{\partial v}{\partial \bar{v}}=\frac{\partial Q}{\partial \bar{v}}=Q_2.$$

Then we have the transformation

$$\bar{x}_1=x_1P_1+x_2Q_1, \quad \bar{x}_2=x_1P_2+x_2Q_2. \tag{22}$$

The modulus of this transformation is denoted by

$$\delta=P_1Q_2-P_2Q_1=\begin{vmatrix} P_1 P_2 \\ Q_1 Q_2 \end{vmatrix}. \tag{23}$$

We wish now to consider forms of expression that are invariant in transformation. When the resulting expression has the same form multiplied by a power of δ it is said to be a *relative* invariant form. When it has the same form, but not multiplied by a power of δ it is said to be an *absolute* invariant.

Elementary Invariant Vector Forms.—When a vector $\bar{\xi}$ is transformed into the vector ξ we have

$$\begin{aligned} \textit{Form (I)} \quad d\bar{\xi} &= \bar{\xi}_1 d\bar{u}+\bar{\xi}_2 d\bar{v}=(\xi_1P_1+\xi_2Q_1)\,d\bar{u}+(\xi_1P_2+\xi_2Q_2)\,d\bar{v} \\ &= \xi_1(P_1d\bar{u}+P_2d\bar{v})+\xi_2(Q_1d\bar{u}+Q_2d\bar{v})=\xi_1du+\xi_2dv=d\xi. \end{aligned}$$

This shows the complete differential to be invariant, and hence we have the complete derivative to be invariant. Hence vector expressions composed of factors that are complete derivatives are invariant. Thus

$$D=-Sax'x'', \quad W=-Saa'x', \quad Sx'x', \text{ \&c.,}$$

are invariant.

$$\begin{aligned} \textit{Form (II)} \quad V\bar{\xi}_1\bar{\xi}_2 &= V(\xi_1P_1+\xi_2Q_1)(\xi_1P_2+\xi_2Q_2) \\ &= (P_1Q_2-P_2Q_1)V\xi_1\xi_2=\delta V\xi_1\xi_2, \end{aligned}$$

which shows this form to be invariant.

Under this form we have the invariants

$$\Delta=-Sax_1x_2=-SaVx_1x_2, \quad \Delta k=-Saa_1a_2=-SaVa_1a_2, \quad Saw_1w_2=SaVw_1w_2.$$

$$\begin{aligned} \textit{Form (III)} \quad S\bar{\xi}_1\bar{x}_2-S\bar{\xi}_2\bar{x}_1 &= S(\xi_1P_1+\xi_2Q_1)(x_1P_2+x_2Q_2) \\ &\quad -S(\xi_1P_2+\xi_2Q_2)(x_1P_1+x_2Q_1)=(P_1Q_2-P_2Q_1)(S\xi_1x_2-S\xi_2x_1) \\ &= \delta(S\xi_1x_2-S\xi_2x_1). \end{aligned}$$

This form is made up of the difference of two scalar terms of which the partial derivatives 1, 2 in the first term are respectively 2, 1 in the second term.

In a similar manner we may show the 1, 2 and 2, 1 form to apply to a large number of invariant expressions. Thus,

$$Sx_1w_2 - Sx_2w_1, \quad Sax_1w_2 - Sax_2w_1, \quad x_1\Delta_2 - x_2\Delta_1, \quad \phi_2x_1 - \phi_1x_2, \text{ \&c.,}$$

are all invariant forms.

Of especial importance under Form (III) is the invariant expression

$$\frac{dK}{dn} = -\frac{1}{\Delta}\{K_2Sx'x_1 - K_1Sx'x_2\}, \quad \text{see (14)}$$

where K represents either a scalar or a vector quantity. By the use of this invariant expression we have, when K is a vector, a number of invariant forms. Thus each factor in

$$Sa\frac{dK}{dn}, \quad Sax'\frac{dK}{dn}, \quad Saw\frac{dK}{dn}, \quad Sx''\frac{dK}{dn}, \quad Sax'\frac{dK'}{dn}, \quad Saw\frac{dK'}{dn},$$

being invariant, the forms themselves are invariant.

Every invariant in differential geometry may be shown to be composed of one or more of the elementary forms (I), (II), or (III). If we can evaluate any of the invariant forms that we can write out from our simple elements in terms of well-known invariant expressions, we will be enabled to evaluate any invariant or covariant expression of differential geometry.

Forsyth has shown in his "Differential Geometry" that all invariants and covariants formed by the use of derivatives below the third order may be expressed in terms of any set of twelve such expressions that may be selected. From the set selected by him he derived eleven absolute invariant expressions as his fundamental set. For the set that we will use in this paper let us take the following by means of which we will evaluate all other invariant forms made by derivatives below the third order:

$D = -Sax'x''$, the geodesic curvature.

$W = -Saa'x'$, the geodesic torsion.

$k = -\frac{1}{\Delta}Saa_1a_2$, the Gaussian curvature where $\Delta = -Sax_1x_2$.

$\beta = \frac{r}{\Delta} = \frac{\sqrt{-Sww}}{\Delta}$, the differential parameter of the first order.

$h = \frac{1}{\Delta}\{Saa_1x_2 - Saa_2x_1\}$, the mean curvature.

$\frac{1}{\rho} = Sa'x' = -Sax''$, the curvature of a normal section.

$$\frac{d\beta}{ds}, \quad \frac{d\beta}{dn}, \quad \frac{dh}{ds}, \quad \frac{dh}{dn}, \quad \frac{d}{ds}\left(\frac{1}{\rho}\right), \quad \frac{d}{dn}\left(\frac{1}{\rho}\right).$$

Between these there exists the relation

$$k=\frac{h}{\rho}-\frac{1}{\rho^2}-W^2,$$

making in reality only eleven absolute invariant forms to be used.

We will now evaluate a number of *invariant forms* and will later show that invariants of differential geometry are composed of those forms. It will be seen that every form that we notice will be invariant by reason of elementary Form (I), Form (II) or Form (III). Hence, while we may evaluate invariant or covariant expressions of differential geometry by means of a set of eleven, we may express all such invariants or covariants in terms of *three elementary type forms.*

II. *Evaluation of Invariant Forms.*

We have defined

$$w=\phi_2 x_1-\phi_1 x_2=rx', \tag{24}$$

which is seen to be invariant by type form III. Also we have

$$Sww=r^2Sx'x'=-r^2, \text{ since } Sx'x'=-1. \tag{25}$$

$$Sx'w'=Sx'(r'x'+rx'')=-r', \text{ since } Sx'x''=0. \tag{26}$$

$$Sww'=rSx'w'=-rr'. \tag{27}$$

Since $Sax_1x_2=-\Delta$ we have

$$Sawx_2=Sa(\phi_2x_1-\phi_1x_2)x_2=-\phi_2\Delta, \quad Sawx_1=Sa(\phi_2x_1-\phi_1x_2)x_1=-\phi_1\Delta.$$

By differentiation we find

$$Saw_1x_2+Sawx_{12}=-\phi_{12}\Delta-\phi_2\Delta_1, \quad Saw_2x_1+Sawx_{12}=-\phi_{12}\Delta-\phi_1\Delta_2.$$

$$\therefore Saw_2x_1-Saw_1x_2=\phi_2\Delta_1-\phi_1\Delta_2=r\Delta'. \tag{28}$$

We have seen that we may write $ru'=\phi_2$ and $rv'=-\phi_1$, and from these we find $r_1u'+ru''=\phi_{12}=-r_2v'-rv''$,

$$\therefore r(u''+v'')=-\{r_1u'+r_2v'\}=-r'. \tag{29}$$

We have seen that

$$\Delta h= Saa_1x_2-Saa_2x_1. \text{ See (20)} \tag{30}$$

$$\Delta k=-Saa_1a_2. \text{ See (10).} \tag{31}$$

$$Sa'a'=-\frac{1}{\Delta}Sa'a'Sax_1x_2$$

$$=-\frac{1}{\Delta}\{Sa'x_1Saa'x_2-Sa'x_2Saa'x_1\}, \text{ since } Saa'=0$$

$$=-\frac{1}{\Delta}\{Sa_1x'Saa'x_2-Sa_2x'Saa'x_1\}, \text{ since } Sa'x_1=Sa_1x', \&c.,$$

$$=-\frac{1}{\Delta}\{Sa'x'(Saa_1x_2-Saa_2x_1)+Sx'x_2Saa'a_1-Sx'x_1Saa'a_2\}$$

$$=-\frac{1}{\Delta}\left\{\frac{\Delta h}{\rho}-Saa_1a_2Sx'(x_2v'+x_1u')\right\}, \text{ since } Saa'a_1=v'Saa_2a_1$$

$$=-\frac{1}{\Delta}\left\{\frac{\Delta h}{\rho}+\Delta kSx'x'\right\}=-\frac{h}{\rho}+k. \tag{32}$$

It is easily seen that the vector $Va'x'$ is parallel to a, the unit vector normal to the surface. Hence we have

$$Wa=Va'x', \text{ since } W=-Saa'x'.$$

$$\therefore W^2=-SVa'x'Va'x'=-Sa'x'Sa'x'+Sa'a'Sx'x'=-\frac{1}{\rho^2}+\frac{h}{\rho}-k. \tag{33}$$

$$WD=Saa'x'Sax'x''$$

$$=-\begin{vmatrix} Saa & Sax' & Sax'' \\ Sa'a & Sa'x' & Sa'x'' \\ Sx'a & Sx'x' & Sx'x'' \end{vmatrix}=-\begin{vmatrix} -1 & 0 & Sax'' \\ 0 & Sa'x' & Sa'x'' \\ 0 & -1 & 0 \end{vmatrix}, \text{ since } Sx'x''=0$$

$$=Sa'x''. \tag{34}$$

$$Saa'x''=-Sx'x'Saa'x''=-Sx'a'Sax'x''-Sx'x''Saa'x', \text{ since } Sx'a=0$$

$$=\frac{D}{\rho}, \text{ since } Sx'x''=0. \tag{35}$$

$$Saa'w'=Saa'(r'x'+rx'')=-r'W+\frac{rD}{\rho}. \tag{36}$$

$$Sx'\frac{dx}{dn}=Sx'Vax'=0. \tag{37}$$

$$Sx'\frac{dw}{dn}=Sx'\left(r\frac{dx'}{dn}+x'\frac{dr}{dn}\right), \text{ since } w=rx'$$

$$=-\frac{dr}{dn}. \tag{38}$$

$$Sx'\frac{da}{dn}=-\frac{1}{\Delta}\{Sx'x_1Sx'a_2-Sx'x_2Sx'a_1\}, \text{ see (14)}$$

$$=-\frac{1}{\Delta}\{Sx'x_1Sa'x_2-Sx'x_2Sa'x_1\}, \text{ since } Sa'x_2=Sx'a_2 \text{ and } Sa'x_1=Sx'a_1$$

$$=-\frac{1}{\Delta}SVx_1x_2Va'x'=-Saa'x', \text{ since } \Delta a=Vx_1x_2$$

$$=W. \tag{39}$$

$$Sa\frac{dw}{dn}=-Sw\frac{da}{dn}, \text{ since } Saw=0$$

$$=-rSx'\frac{da}{dn}=-rW. \tag{40}$$

$$Sa'\frac{dx}{dn} = \quad Sa'Vax' = -Saa'x' = W. \tag{41}$$

$$\begin{aligned} Sax'\frac{dw}{dn} &= -\frac{1}{\Delta}\{Sx'x_1Sax'w_2 - Sx'x_2Sax'w_1\} \\ &= -\frac{1}{\Delta}\{Sx'x'(Sax_1w_2 - Sax_2w_1) + Sx'w_2Sax'x_1 - Sx'w_1Sax'x_2\} \\ &= -\frac{1}{\Delta}\{r\Delta' + \Delta Sx'(w_2v' + w_1u')\} = -\frac{1}{\Delta}(r\Delta' + \Delta Sx'w') \\ &= -\frac{1}{\Delta}(r\Delta' - \Delta r') = \frac{\Delta r' - r\Delta'}{\Delta} = \Delta\left(\frac{r}{\Delta}\right)' = \Delta\beta', \end{aligned}$$

since we define $\beta = \frac{r}{\Delta}$. (42)

$$Saw\frac{dw}{dn} = rSax'\frac{dw}{dn} = r\Delta\beta', \text{ since } w = rx'. \tag{43}$$

$$\begin{aligned} Sax'\frac{dx'}{dn} &= Sax'\frac{d}{dn}\left(\frac{w}{r}\right) = \frac{1}{r}Sax'\frac{dw}{dn}, \text{ since } \frac{d}{du}\left(\frac{w}{r}\right) = \frac{1}{r}\frac{dw}{dn} + w\frac{d}{dn}\left(\frac{1}{r}\right) \\ &= \frac{1}{r}\Delta\beta' = \frac{\beta'}{\beta}. \end{aligned} \tag{44}$$

$$Sax'\frac{dx}{dn} = SVax'Vax' = -SaaSx'x' = -1. \tag{45}$$

$$Saw'\frac{dx}{dn} = -Sx'x'Saw'\frac{dx}{dn} = -Sx'w'Sax'\frac{dx}{dn} = -r'. \tag{46}$$

$$\begin{aligned} Saw'\frac{dw}{dn} &= -Sx'x'Saw'\frac{dw}{dn} = -Sx'w'Sax'\frac{dw}{dn} - Sx'\frac{dw}{dn}Saw'x' \\ &= +r'\Delta\beta' - rDSx'\frac{d}{dn}(rx'), \text{ since } Sax'w' = Sax'(r'x' + rx'') = -rD \\ &= +r'\Delta\beta' + rD\frac{dr}{dn}. \end{aligned} \tag{47}$$

$$Saw'\frac{dx'}{dn} = -Sx'x'Saw'\frac{dx'}{dn} = -Sx'w'Sax'\frac{dx'}{dn} - Sx'\frac{dx'}{dn}Saw'x' = \frac{\beta'}{\beta}\frac{r'}{1}. \tag{48}$$

$$Saa'\frac{dx'}{dn} = -Sx'x'Saa'\frac{dx'}{dn} = -Sx'a'Sax'\frac{dx'}{dn} - Sx'\frac{dx'}{dn}Saa'x' = -\frac{1}{\rho}\frac{\beta'}{\beta}. \tag{49}$$

$$\begin{aligned} Saa'\frac{dw}{dn} &= -Sx'x'Saa'\frac{dw}{dn} \\ &= -Sx'a'Sax'\frac{dw}{dn} - Sx'\frac{dw}{dn}Saa'x' = -\frac{1}{\rho}\Delta\beta' - W\frac{dr}{dn}. \end{aligned} \tag{50}$$

$$Saa'\frac{dx}{dn}=SVaa'Vax'=-SaaSa'x'=\frac{1}{\rho}. \tag{51}$$

$$\begin{aligned}Sax'\frac{da}{dn}&=-\frac{1}{\Delta}\{Sx'x_1Sax'a_2-Sx'x_2Sax'a_1\}\\&=-\frac{1}{\Delta}\{Sx'x'(Sax_1a_2-Sax_2a_1)+Sx'a_2Sax'x_1-Sx'a_1Sax'x_2\}\\&=-\frac{1}{\Delta}\{-\Delta h+\Delta Sx'(a_2v'+a_1u')\}=h-\frac{1}{\rho}.\end{aligned} \tag{52}$$

$$\begin{aligned}Saw'\frac{da}{dn}&=-Sx'x'Saw'\frac{da}{dn}\\&=-Sx'w'Sax'\frac{da}{dn}-Sx'\frac{da}{dn}Saw'x'=r'\left(h-\frac{1}{\rho}\right)-rDW.\end{aligned} \tag{53}$$

$$\begin{aligned}Saa'\frac{da}{dn}&=-Sx'x'Saa'\frac{da}{dn}=-Sx'a'Sax'\frac{da}{dn}-Sx'\frac{da}{dn}Saa'x'\\&=-\frac{1}{\rho}\left(h-\frac{1}{\rho}\right)+W^2=-k.\quad\text{See (33).}\end{aligned} \tag{54}$$

$$Sax''\frac{da}{dn}=-Sx'x'Sax''\frac{da}{dn}=-Sx'\frac{da}{dn}Sax''x'=-WD. \tag{55}$$

$$\begin{aligned}Saw_1w_2&=-Sx'x'Saw_1w_2=-Sx'(x_1u'+x_2v')Saw_1w_2\\&=-\{Sx'x_1Saw'w_2-Sx'x_2Saw'w_1\}\\&=\Delta Saw'\frac{dw}{dn}=\Delta^2r'\beta'+\Delta rD\frac{dr}{dn}.\quad\text{See (47).}\end{aligned} \tag{56}$$

$$\begin{aligned}Sx''\frac{da}{dn}&=-\frac{1}{W}Sx''\frac{da}{dn}Saa'x'\\&=+\frac{1}{W}Sx''a'Sax'\frac{da}{dn},\ \text{since } Sa'x'\frac{da}{dn}=0 \text{ and } Sx''x'=0\\&=D\left(h-\frac{1}{\rho}\right).\quad\text{See (34).}\end{aligned} \tag{57}$$

$$\begin{aligned}Sa'\frac{dw}{dn}&=-\frac{1}{rD}Sa'\frac{dw}{dn}Sax'w'=-\frac{1}{rD}\left\{-Sa'x'Saw'\frac{dw}{dn}+Sa'w'Sax'\frac{dw}{dn}\right\}\\&=-\frac{1}{rD}\left\{-\frac{1}{\rho}\left(r'\Delta\beta'+rD\frac{dr}{du}\right)+\Delta\beta'\left(\frac{r'}{\rho}+rDW\right)\right\}\\&=-\frac{1}{rD}\left\{-rD\frac{dr}{du}+rDW\Delta\beta'\right\}=\frac{dr}{du}-\Delta W\beta'.\end{aligned} \tag{58}$$

$$\begin{aligned}Sa'\frac{dx'}{dn}&=-\frac{1}{rD}Sa'\frac{dx'}{dn}Sax'w'=-\frac{1}{rD}\left\{-Sa'x'Saw'\frac{dx'}{dn}+Sa'w'Sax'\frac{dx'}{dn}\right\}\\&=-\frac{1}{rD}\left\{-\frac{1}{\rho}\frac{\beta'}{\beta}r'+\frac{\beta'}{\beta}\left(\frac{r'}{\rho}+rWD\right)\right\}=-W\frac{\beta'}{\beta}.\end{aligned} \tag{59}$$

$$Sw\frac{da'}{dn}=\frac{d}{dn}\left(\frac{r}{\rho}\right)-Sa'\frac{dw}{dn},\text{ since }Sa'w=\frac{r}{\rho}$$

$$=\frac{d}{dn}\left(\frac{r}{\rho}\right)-Sa'\left(r\frac{dx'}{dn}+x'\frac{dr}{dn}\right),\text{ since }w=rx'$$

$$=\frac{d}{dn}\left(\frac{r}{\rho}\right)+rW\frac{\beta'}{\beta}-\frac{1}{\rho}\frac{dr}{dn}$$

$$=r\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{1}{\rho}\frac{dr}{dn}+rW\frac{\beta'}{\beta}-\frac{1}{\rho}\frac{dr}{dn}=r\frac{d}{dn}\left(\frac{1}{\rho}\right)+rW\frac{\beta'}{\beta}. \quad (60)$$

$$Sx'\frac{da'}{dn}=\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{\Delta W\beta'}{r}. \quad (61)$$

$$Sx''\frac{dw}{dn}=-\frac{1}{W}Sx''\frac{dw}{dn}Saa'x'=-\frac{1}{W}\left\{Sx''aSa'x'\frac{dw}{dn}-Sx''a'Sax'\frac{dw}{dn}\right\}$$

$$=-\frac{1}{W}\left\{-\frac{1}{\rho}WSa\frac{dw}{dn}-WD\Delta\beta'\right\},\text{ since }Wa=Va'x'$$

$$=-\frac{r}{\rho}W+D\Delta\beta'. \quad (62)$$

$$Sa''\frac{dx}{dn}=-Saa''x',\text{ since }\frac{dx}{dn}=Vax'$$

$$=-Saa''x'-Saa'x''+Saa'x''=W'+\frac{D}{\rho}.\quad\text{See (35).} \quad (63)$$

$$Sa''\frac{dx}{dn}=\frac{1}{\Delta}\{Sx'x_2Sa''x_1-Sx'x_1Sa''x_2\}$$

$$=\frac{1}{\Delta}\{Sx'x_2(Sa''x_1-Sx''a_1)-Sx'x_1(Sa''x_2-Sx''a_2)\}+Sx''\frac{da}{dn}$$

$$=-\frac{1}{\Delta}[Sx'x_2\{Sa'(x_1)'-Sx'(a_1)'\}-Sx'x_1\{Sa'(x_2)'-Sx'(a_2)'\}]$$

$$+Sx''\frac{da}{dn},\text{ since }Sx'a_1=Sa'x_1$$

$$=-\frac{1}{\Delta}[Sx'x_2\{Sa'(x')_1-Sx'(a')_1\}-Sx'x_1\{Sa'(x')_2-Sx'(a')_2\}]$$

$$+Sx''\frac{da}{dn}$$

$$=\frac{1}{\Delta}[Sx'x_2\{Sa'(x')_1+Sx'(a')_1\}-Sx'x_1\{Sa'(x')_2+Sx'(a')_2\}]$$

$$-\frac{2}{\Delta}[Sx'x_2Sa'(x')_1-Sx'x_1Sa'(x')_2]+Sx''\frac{da}{dn}=\frac{d}{dn}\left(\frac{1}{\rho}\right)$$

$$-\frac{2}{\Delta}Sa'\frac{dx'}{dn}+Sx''\frac{da}{dn}=\frac{d}{dn}\left(\frac{1}{\rho}\right)+2W\frac{\beta'}{\beta}+D\left(h-\frac{1}{\rho}\right). \quad (64)$$

See (59) and (57).

$$W' = -Saa''x' - Saa'x'' = Sa''Vax' - Saa'x'' = Sa''\frac{dx}{dn} - Saa'x''$$
$$= \frac{d}{dn}\left(\frac{1}{\rho}\right) + 2W\frac{\beta'}{\beta} + D\left(h - \frac{1}{\rho}\right) - \frac{D}{\rho}. \quad \text{See (64) and (35).}$$
$$= \frac{d}{dn}\left(\frac{1}{\rho}\right) + 2W\frac{\beta'}{\beta} + D\left(h - \frac{2}{\rho}\right). \tag{65}$$

$$Saa'a'' = -Sx'x'Saa'a'' = -Sx'a'Sax'a'' - Sx'a''Saa'x'$$
$$= -\frac{1}{\rho}(Sax''a' + Sax'a'') + \frac{1}{\rho}Sax''a' + WSa''x'$$
$$= -\frac{W'}{\rho} - \frac{D}{\rho^2} + W(K + WD), \text{ since } K = Sa''x' - Sa'x'' = Sa''x' - WD$$
$$= -\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right) - D\left(\frac{h}{\rho} - \frac{1}{\rho^2} - W^2\right) - \frac{2}{\rho}\frac{W\beta'}{\beta} + WK$$
$$= -\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right) - Dk - \frac{2W\beta'}{\beta} + WK. \tag{66}$$

By the use of similar methods we find

$$Sx_2w_1 - Sx_1w_2 = -\Delta\left(rD - \frac{dr}{dn}\right). \tag{67}$$

$$Sx'(\Delta_1a_2 - \Delta_2a_1) = \Delta\left(W\Delta' - \frac{1}{\rho}\frac{d\Delta}{dn}\right). \tag{68}$$

$$Saa_1w_2 - Saa_2w_1 = -\Delta W\left(rD + \frac{dr}{dn}\right) - \frac{\Delta}{\rho}\Delta'\beta - \frac{2\Delta^2\beta'}{\rho} + r'\Delta h. \tag{69}$$

$$Sax'(\Delta_1w_2 - \Delta_2w_1) = -r\Delta'^2 + \Delta rD\frac{d\Delta}{dn} + \Delta\Delta'r'. \tag{70}$$

$$Sax'(\Delta_1a_2 - \Delta_2a_1) = \Delta\Delta'h - \Delta W\frac{d\Delta}{dn} - \frac{\Delta}{\rho}\Delta'. \tag{71}$$

$$Sa_1w_2 - Sa_2w_1 = -\Delta rD\left(h - \frac{1}{\rho}\right) + \frac{\Delta}{\rho}\frac{dr}{dn} - 2\Delta^2W\beta' - r\Delta'W. \tag{72}$$

$$Saa_1(x')_2 - Saa_2(x')_1 = -\frac{\Delta\beta'}{\rho\beta} - \Delta WD. \tag{73}$$

$$Sa(a')_1x_2 - Sa(a')_2x_1 = \Delta h' - \frac{\Delta\beta'}{\beta}\left(h - \frac{1}{\rho}\right) + \Delta WD. \tag{74}$$

Each form evaluated above is seen to be invariant by virtue of the type forms I, II or III, of which it is composed.

III. *Invariants of Differential Geometry.*

In what follows invariants or covariants of differential geometry will be seen to be composed of one or more invariant forms, and hence may be expressed in terms of one or more of the three type forms I, II and III. It will thus be seen, also, that the invariants and covariants may be evaluated by the use of the results obtained in the previous section.

In making the evaluations of invariant forms the derivatives of r and Δ were used in several instances. The reader will notice that in combining two or more of these forms to express an invariant or covariant in differential geometry that the derivatives of r and Δ are eliminated by the use of the relation $r=\Delta\beta$.

In what follows the invariancy of an expression will be indicated by giving at the right the type form or forms of which it is composed.

The quadratic form

$$\theta=E\phi_2^2-2F\phi_2\phi_1+G\phi_1^2$$

was shown in (25) to be given by

$$\theta=-Sww=-S(\phi_2x_1-\phi_1x_2)(\phi_2x_1-\phi_1x_2) \quad \text{(Form III)}. \tag{a}$$

From this we obtain the absolute invariant

$$\frac{\theta}{\Delta^2}=-\frac{Sww}{\Delta^2}=\beta^2, \tag{a'}$$

where β^2 is the differential parameter of the first order.

Again we have the quadratic form

$$\begin{aligned}\chi&=L\phi_2^2-2M\phi_2\phi_1+N\phi_1^2=S(\phi_2x_1-\phi_1x_2)(\phi_2a_1-\phi_1a_2) \quad \text{(Form III)}.\\ &=Sw\bar{\omega}, \text{ since } \bar{\omega}=\phi_2a_1-\phi_1a_2\\ &=r^2Sx'a', \text{ since } w=rx' \text{ and } \bar{\omega}=ra'=\frac{r^2}{\rho}. \quad \text{See (13).}\end{aligned} \tag{b}$$

From this we obtain the absolute invariant form

$$\frac{\chi}{\Delta^2}=\frac{\beta^2}{\rho}, \text{ since } \beta=\frac{r}{\Delta}. \tag{b'}$$

We have seen also in (10),

$$\begin{aligned}k=&\frac{1}{\Delta^2}(LN-M^2)=\frac{1}{\Delta^2}(Sa_1x_1Sa_2x_1-Sa_2x_1Sa_1x_2)\\ &=-\frac{1}{\Delta^2}SVa_1a_2Vx_1x_2 \quad \text{(Form II)}\\ &=-\frac{1}{\Delta}Saa_1a_2.\end{aligned} \tag{c}$$

In differential geometry this absolute invariant is often written

$$k=\frac{1}{\rho_1\rho_2}. \qquad (c')$$

After omitting the numerical factors the functional determinant of θ and χ may be written

$$J_{\theta\chi}=-\begin{vmatrix} Sx_1w, & Sx_2w \\ Sa_1w, & Sa_2w \end{vmatrix}=-r^2\{Sx_1x'Sx'a_2-Sx_2x'Sx'a_1\}$$

$$= \Delta r^2 Sx'\frac{da}{dn}=\Delta r^2 W. \quad \text{See (39).} \qquad (d)$$

From this we find the absolute invariant

$$\frac{J_{\theta\chi}}{\Delta^3}=\beta^2 W. \qquad (d')$$

We have the quadratic

$$d=A\phi_2^2-2B\phi_2\phi_1+C\phi_1^2=r^2(Au'^2+2Bu'v'+Cv^2),$$

where

$$\Delta A=\Delta\phi_{11}+Sawx_{11}, \quad \Delta B=\Delta\phi_{12}+Sawx_{12}, \quad \Delta C=\Delta\phi_{22}+Sawx_{22}.$$

$$\therefore\ d=\frac{r^2}{\Delta}\{\Delta(\phi_{11}u'^2+2\phi_{12}u'v'+\phi_{22}v'^2)+Saw(x_{11}u'^2+2x_{12}u'v'+x_{22}v'^2)\}$$

$$=\frac{r^2}{\Delta}\{-\Delta(\phi_1u''+\phi_2v'')+Saw(x''-x_1u''-x_2v'')\}$$

$$=\frac{r^2}{\Delta}\{-\Delta(\phi_1u''+\phi_2v'')+Sawx''+\Delta(\phi_1u''+\phi_2v'')\}$$

$$=\frac{r^3}{\Delta}Sax'x'' \quad \text{(Form I).} \qquad (e)$$

In this last form d is seen to be invariant. We have also the absolute invariant form

$$\frac{d}{\Delta^2}=-\beta^3 D, \text{ since } D=-Sax'x''. \qquad (e')$$

More useful expressions for A, B and C may be found as follows:

$$Sawx_1=Sa(\phi_2x_1-\phi_1x_2)x_1=-\phi_1\Delta, \quad Sawx_2=Sa(\phi_2x_1-\phi_1x_2)x_2=-\phi_2\Delta.$$

By differentiation we have,

$$Sawx_{11}+Saw_1x_1=-\phi_1\Delta_1-\phi_{11}\Delta, \quad Sawx_{12}+Saw_2x_1=-\phi_1\Delta_2-\phi_{12}\Delta,$$
$$Sawx_{12}+Saw_1x_2=-\phi_2\Delta_1-\phi_{12}\Delta, \quad Sawx_{22}+Saw_2x_2=-\phi_2\Delta_2-\phi_{22}\Delta,$$

and by the use of these relations we find

$$\Delta A=-\phi_1\Delta_1-Saw_1x_1, \quad \Delta B=-\phi_1\Delta_2-Saw_2x_1=-\phi_2\Delta_1-Saw_1x_2,$$
$$\Delta C=-\phi_2\Delta_2-Saw_2x_2.$$

The intermediate invariant of θ and d may be written

$$I=AG-2BF+CE=\frac{1}{\Delta}\{Sx_2x_2(\Delta_1\phi_1+Saw_1x_1)-Sx_1x_2(\phi_2\Delta_1+Saw_1x_2)$$
$$-Sx_1x_2(\phi_1\Delta_2+Saw_2x_1)+Sx_1x_1(\phi_2\Delta_2+Saw_2x_2)\}$$
$$=\frac{1}{\Delta}\{\Delta_1Sx_2(\phi_1x_2-\phi_2x_1)-\Delta_2Sx_1(\phi_1x_2-\phi_2x_1)+Sx_2w_1Sax_2x_1$$
$$+Sx_2x_1Saw_1x_2-Sx_1x_2Saw_1x_2-Sx_1w_2Sax_2x_1+Sx_1x_1Saw_2x_2-Sx_1x_1Saw_2x_2\}$$
$$=\frac{1}{\Delta}\{\Delta_2Sx_1w-\Delta_1Sx_2w+\Delta(Sx_2w_1-Sx_1w_2)\}\quad\text{(Form III)}$$
$$=\frac{1}{\Delta}\left\{-r\Delta\frac{d\Delta}{dn}-\Delta^2\left(rD-\frac{dr}{dn}\right)\right\}.\quad\text{See (14) and (67)}$$
$$=-\Delta rD+\Delta\frac{dr}{dn}-\beta\frac{d\Delta}{dn}\Delta=\Delta^2\frac{d\beta}{dn}-\beta\Delta^2D,\ \text{since } r=\Delta\beta$$
$$=\Delta^2\left(\frac{d\beta}{dn}-\beta D\right),\qquad\text{(f)}$$

and from this we have the absolute invariant

$$\frac{I}{\Delta^2}=\frac{d\beta}{dn}-\beta D.\qquad\text{(f')}$$

It is well to notice that the above invariant may be written

$$I=\Delta\left\{Sx_2\left(\frac{w}{\Delta}\right)_1-Sx_1\left(\frac{w}{\Delta}\right)_2\right\}\quad\text{(Form III)}$$

or $$\frac{I}{\Delta^2}=\frac{1}{\Delta}\left\{\left(\frac{Sx_2w}{\Delta}\right)_1-\left(\frac{Sx_1w}{\Delta}\right)_2\right\}=\frac{1}{\Delta}\left\{\frac{\partial}{\partial u}\left(\frac{G\phi_1-F\phi_2}{\Delta}\right)-\frac{\partial}{\partial v}\left(\frac{E\phi_2-F\phi_1}{\Delta}\right)\right\},$$

which is the *differential parameter of the second order.*

By omitting the numerical factors, the Jacobian of θ and d may be written:

$$J_{\theta d}=-\begin{vmatrix} Sx_1w, & Sx_2w \\ (A\phi_2-B\phi_1), & (B\phi_2-C\phi_1)\end{vmatrix}$$
$$=\frac{1}{\Delta}\begin{vmatrix} Sx_1w, & \phi_2(\phi_1\Delta_1+Saw_1x_1)-\phi_1(\phi_2\Delta_1+Saw_1x_2), \\ Sx_2w, & \phi_2(\phi_1\Delta_2+Saw_2x_1)-\phi_1(\phi_2\Delta_2+Saw_2x_2)\end{vmatrix}$$
$$=\frac{1}{\Delta}\begin{vmatrix} Sx_1w & Sx_2w \\ Saw_1w & Saw_2w\end{vmatrix},\ \text{since } w=\phi_2x_1-\phi_1x_2$$
$$=-\frac{r^2}{\Delta}\{Sx_1x'Sax'w_2-Sx_2x'Sax'w_1\}\quad\text{(Form III)}$$
$$=r^2Sax'\frac{dw}{du}=r^2\Delta\beta'.\quad\text{See (42).}\qquad\text{(g)}$$

$$\therefore\ \frac{J\theta d}{\Delta^3}=\beta^2\beta'.\qquad\text{(g')}$$

The cubic

$$\overline{K}=P\phi_2^3-3Q\phi_2^2\phi_1+3R\phi_2\phi_1^2-S\phi_1^3=r^3(Pu'^3+3Qu'^2v'+3Ru'v'^2+Sv'^3)$$
$$=r^3(Sa''x'-Sa'x'')\quad\text{(Form I).}\qquad\text{(h)}$$

is seen in this last form to be invariant.

This cubic may be expressed in terms of well-known invariants as follows:

$$\begin{aligned}\overline{K}&=r^3(Sa''x'+Sa'x''-2Sa'x'')\\&=r^3\left\{\frac{d}{ds}(Sa'x')-2Sa'x''\right\}=r^3\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-2WD\right\},\end{aligned}$$

and from this we find

$$\frac{\overline{K}}{\Delta^3}=\beta^3\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-2WD\right\}. \tag{h'}$$

From the cubic $\overline{K}$, let us take

$$\begin{aligned}K&=Sa''x'-Sx''a'=S(a_{11}u'^2+2a_{12}u'v'+a_{22}v'^2)x'-S(x_{11}u'^2+2x_{12}u'v'+x_{22}v'^2)a'\\&=u'^3\{Sa_{11}x_1-Sx_{11}a_1\}+u'^2v'\{(Sa_{11}x_2-Sx_{11}a_2)+2(Sa_{12}x_1-Sx_{12}a_1)\}\\&\qquad+u'v'^2\{2(Sa_{12}x_2-Sx_{12}a_2)+(Sa_{22}x_1-Sx_{22}a_1)\}+v'^3\{Sa_{22}x_2-Sx_{22}a_2\}.\end{aligned}$$

Now we have $Sa_1x_2=Sx_1a_2$, and by differentiation we find

$$Sa_{11}x_2-Sx_{11}a_2=Sa_{12}x_1-Sx_{12}a_1,\quad Sa_{12}x_2-Sx_{12}a_2=Sa_{22}x_1-Sx_{22}a_1.$$

If now we write

$$K=Pu'^3+3Qu'^2v'+3Ru'v'^2+Sv'^3,$$

we have from the above,

$$\begin{aligned}P&=Sa_{11}x_1-Sx_{11}a_1,\quad Q=Sa_{12}x_1-Sx_{12}a_1=Sa_{11}x_2-Sx_{11}a_2,\\R&=Sa_{12}x_2-Sx_{12}a_2=Sa_{22}x_1-Sx_{22}a_1,\quad S=Sa_{22}x_2-Sx_{22}a_2.\end{aligned}$$

Let us now denote

$$K_1=Pu'^2+2Qu'v'+Rv'^2,\quad K_2=Qu'^2+2Ru'v'+Sv'^2.$$

Then we find

$$\begin{aligned}K_1&=u'(Pu'+Qv')+v'(Qu'+Rv')\\&=u'\{S(a_1)'x_1-S(x_1)'a_1\}+v'\{S(a_2)'x_1-S(x_2)'a_1\}\\&=u'\{S(a')_1x_1-S(x')_1a_1\}+v'\{S(a')_2x_1-S(x')_2a_1\}\\&=Sx_1\{(a')_1u'+(a')_2v'\}-Sa_1\{(x')_1u'+(x')_2v'\}\\&=Sa''x_1-Sx''a_1,\end{aligned}$$

and in like manner $K_2=Sa''x_2-Sx''a_2$.

The expression

$$\begin{aligned}\sigma=&\ \{E^2S-3EFR+(EG+2F^2)Q-FGP\}\phi_2\\&\quad-\{EFS-(EG+2F^2)R+3FGQ-G^2P\}\phi_1\\=&\ \{E(ES-2FR+GQ)-F(ER-2FQ+GP)\}\phi_2\\&\quad-\{F(ES-2FR+GQ)-G(ER-2FQ+GP)\}\phi_1\\=&\ \Delta^2\{(Eh_2-Fh_1)\phi_2-(Fh_2-Gh_1)\phi_1\}\quad\text{See (21)}\\=&-\Delta^2\{h_2Sx_1(\phi_2x_1-\phi_1x_2)-h_1Sx_2(\phi_2x_1-\phi_1x_2)\}\\=&-\Delta^2r\{h_2Sx_1x'-h_1Sx_2x'\}\quad\text{(Form III)}\\=&\ \Delta^2r\frac{dh}{dn},\end{aligned} \tag{i}$$

is seen in the last two forms to be invariant.

From this we easily derive the absolute invariant form,

$$\frac{\sigma}{\Delta^4}=\beta\frac{dh}{dn}. \tag{i'}$$

Also we have

$$\begin{aligned}\sigma' &= (ER-2FQ-GP)\phi_2-(ES-2FR+GQ)\phi_1\\ &=\Delta^2(h_1\phi_2-h_2\phi_1) \quad \text{See (21)} \quad \text{(Form III)}\\ &=r\Delta^2h'\end{aligned} \tag{j}$$

which is seen to be invariant. It may also be written

$$\frac{\sigma'}{\Delta^3}=\beta h', \tag{j'}$$

as an absolute invariant.

The Jacobian of the cubic K and the quadratic θ may be written:

$$J_{K\theta}=-r^3\begin{vmatrix} Sx'x_1 & Sx'x_2\\ Sa''x_1-Sx''a_1 & Sa''x_2-Sx''a_2\end{vmatrix}=\Delta r^3\left\{Sa''\frac{dx}{dn}-Sx''\frac{da}{dn}\right\}$$

(Forms I and III)

$$=\Delta r^3\left\{\frac{d}{dn}\left(\frac{1}{\rho}\right)+2\frac{W\Delta\beta'}{r}+D\left(h-\frac{1}{\rho}\right)-D\left(h-\frac{1}{\rho}\right)\right\}$$

See (57) and (64)

$$=\Delta r^3\left\{\frac{d}{dn}\left(\frac{1}{\rho}\right)+2\frac{W\beta'}{\beta}\right\}. \tag{k}$$

From this we have the absolute invariant form,

$$\frac{J_{K\theta}}{\Delta^4}=\beta^3\frac{d}{dn}\left(\frac{1}{\rho}\right)+2W\beta'\beta^2. \tag{k'}$$

As a special case of the quadratic

$$-Sx'x'=Eu'^2+2Fu'v'+Gv'^2$$

and the cubic $K=Pu'^3+3Qu'^2v'+3Ru'v'^2+Sv'^3=Sa''x'-Sx''a'$,

we may write $$J_{23}=\frac{\Delta}{\beta}\left\{\beta\frac{d}{dn}\left(\frac{1}{\rho}\right)+2W\beta'\right\}.$$

The Jacobian of P and χ may be written

$$J_{P\chi}=6r^3\begin{vmatrix} -Sa_1x' & Sa_2x'\\ (Sa''x_1-Sx''a)_1 & -(Sa''x_2-Sx''a_2)\end{vmatrix}$$

$$=6r^3\begin{vmatrix} Sa'x_1 & Sa'x_2\\ Sa''x_1 & Sa''x_2\end{vmatrix}-6r^3\begin{vmatrix} Sa_1x' & Sa_2x'\\ Sx''a_1 & Sx''a_2\end{vmatrix}, \text{ since } Sa_1x'=Sa'x_1$$

$$=-6r^3\{SVx_1x_2Va'a''-SVa_1a_2Vx'x''\}=-6\Delta r^3\{Saa'a''-kSax'x''\} \quad \text{(Form I)}$$

$$=+6\Delta r^3\left\{\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right)+Dk+\frac{2W\beta'}{\rho\beta}-WP-Dk\right\} \quad \text{See (66) and (16)}$$

$$=6\Delta^3\left\{\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{2W\beta'}{\rho\beta}-WP\right\}. \tag{l}$$

$$\therefore \frac{J_{P\chi}}{\Delta^4}=6\beta^3\left\{\frac{1}{\rho}\frac{d}{dn}\left(\frac{1}{\rho}\right)+\frac{2W\beta'}{\rho\beta}-WP\right\}. \tag{l'}$$

The discriminant of the quadratic d may be given

$$\begin{aligned}\Delta^2(AC-B^2) &= (\phi_1\Delta_1+Saw_1x_1)(\phi_2\Delta_2+Saw_2x_2)-(\phi_1\Delta_2+Saw_2x_1)(\phi_2\Delta_1+Saw_1x_2)\\ &= \Delta_1 Saw_2(\phi_1x_2-\phi_2x_1)+\Delta_2 Saw_1(\phi_2x_1-\phi_1x_2)+Saw_1x_1Saw_2x_2\\ &\quad -Saw_2x_1Saw_1x_2 = Saw(\Delta_1w_2-\Delta_2w_1)+Saw_1\{x_1Saw_2x_2-x_2Saw_2x_1\}\\ &= Saw(\Delta_1w_2-\Delta_2w_1)+Saw_1\{w_2Sax_1x_2+x_2Saw_2x_1-x_2Saw_2x_1\}\\ &= Saw(\Delta_1w_2-\Delta_2w_1)-\Delta Saw_1w_2 \quad \text{(Forms II and III)}\\ &= -r^2\Delta'^2+\Delta r^2D\frac{d\Delta}{du}+r\Delta\Delta'r'-\Delta^3r'\beta'-\Delta^2\frac{dr}{du}rD\end{aligned}$$

See (70) and (56)

$$\begin{aligned}&= -\Delta^2rD\left(\frac{dr}{du}-\beta\frac{d\Delta}{du}\right)-r\Delta\beta\Delta'^2+\Delta r\Delta'(\Delta\beta'+\Delta'\beta)-\Delta^3r'\beta'\\ &= -\Delta^2rD\left(\frac{dr}{du}-\beta\frac{d\Delta}{du}\right)+\Delta^3\beta'(\beta\Delta'-r')\\ &= -\Delta^2rD\left(\frac{dr}{dn}-\beta\frac{d\Delta}{dn}\right)-\Delta^4\beta'^2 = -\Delta^3rD\frac{d\beta}{dn}-\Delta^4\beta'^2. \qquad \text{(m)}\end{aligned}$$

$$\therefore \frac{1}{\Delta^2}(AC-B^2) = -\beta'^2-\beta D\frac{d\beta}{dn}. \qquad \text{(m')}$$

The intermediate invariant of d and χ is given by

$$\begin{aligned}NA-2MB+LC &= -\frac{1}{\Delta}[\{Sa_2x_2(\Delta_1\phi_1+Saw_1x_1)-Sa_2x_1(\Delta_1\phi_2+Saw_1x_2)\}\\ &\quad -\{Sa_1x_2(\Delta_2\phi_1+Saw_2x_1)-Sa_1x_1(\Delta_2\phi_2+Saw_2x_2)\}]\\ &= -\frac{1}{\Delta}[\Delta_1Sa_2(\phi_1x_2-\phi_2x_1)-\Delta_2Sa_1(\phi_1x_2-\phi_2x_1)\\ &\quad +Sa_2x_2Saw_1x_1-Sa_2x_1Saw_1x_2-Sa_1x_2Saw_2x_1+Sa_1x_1Saw_2x_2]\\ &= -\frac{1}{\Delta}\{(\Delta_2Swa_1-\Delta_1Swa_2)+\Delta\{Sa_2w_1-Sa_1w_2\}\} \quad \text{(Form III)}\\ &= \beta\Delta\left(W\Delta'-\frac{1}{\rho}\frac{d\Delta}{dn}\right)-\Delta rD\left(h-\frac{1}{\rho}\right)+\frac{\Delta}{\rho}\frac{dr}{dn}\end{aligned}$$

$$-2W\Delta^2\beta'-\Delta'rW \quad \text{See (68) and (72)}$$

$$= \frac{\Delta^2}{\rho}\frac{d\beta}{dn}-\Delta^2\beta D\left(h-\frac{1}{\rho}\right)-2W\Delta^2\beta'. \qquad \text{(n)}$$

$$\therefore \frac{1}{\Delta^2}\left\{NA-2MB+LC\right\} = \frac{1}{\rho}\frac{d\beta}{dn}-\beta D\left(h-\frac{1}{\rho}\right)-2W\beta'. \qquad \text{(n')}$$

It is well to notice that the above gives the form

$$NA-2MB+LC = \Delta\left\{Sa_1\left(\frac{w}{\Delta}\right)_2-Sa_2\left(\frac{w}{\Delta}\right)_1\right\} \quad \text{(Form III)}.$$

We have

$$\begin{vmatrix} E & F & G \\ L & M & N \\ A & B & C \end{vmatrix} = C(EM-FL)+B(LG-FM)+B(FM-EN)+A(FN-MG).$$

Now we find

$$\begin{aligned} EM-FL &= -[Sx_1x_1Sa_1x_2 - Sx_1x_2Sa_1x_1] = -SVx_1x_2Va_1x_1 = -\Delta Saa_1x_1, \\ LG-FM &= -[Sx_2x_2Sa_1x_1 - Sx_1x_2Sa_1x_2] = \quad SVx_1x_2Va_1x_2 = \quad \Delta Saa_1x_2, \\ FM-EN &= -[Sx_1x_2Sa_2x_1 - Sx_1x_1Sa_2x_2] = \quad SVx_1x_2Va_2x_1 = \quad \Delta Saa_2x_1, \\ FN-MG &= -[Sx_1x_2Sa_2x_2 - Sa_2x_1Sx_2x_2] = -SVx_1x_2Va_2x_2 = -\Delta Saa_2x_2. \end{aligned}$$

$$\begin{aligned} \therefore \begin{vmatrix} E & F & G \\ L & M & N \\ A & B & C \end{vmatrix} &= Saa_1\{x_1(\phi_2\Delta_2+Saw_2x_2)-x_2(\phi_1\Delta_2+Saw_2x_1)\} \\ &\quad -Saa_2\{x_1(\phi_2\Delta_1+Saw_1x_2)-x_2(\phi_1\Delta_1+Saw_1x_1)\} \\ &= \Delta_2 Saa_1w-\Delta_1 Saa_2w+Saa_1\{w_2Sax_1x_2+x_2Saw_2x_1-x_2Saw_2x_1\} \\ &\quad -Saa_2\{w_1Sax_1x_2+x_2Saw_1x_1-x_2Saw_2x_1\} \\ &= Saw(\Delta_1a_2-\Delta_2a_1)+\Delta(Saa_2w_1-Saa_1w_2) \quad \text{(Form III)} \\ &= r\Delta'\Delta h - r\Delta W\frac{d\Delta}{dn} - r\frac{\Delta}{\rho}\Delta' + \Delta^2 W\left(rD+\frac{dr}{dn}\right) \\ &\qquad + \frac{\Delta^2}{\rho}\beta\Delta' + \frac{2\Delta^3\beta'}{\rho} - r'\Delta^2 h \quad \text{See (69) and (71)} \\ &= \Delta^2 h(\beta\Delta'-r') - \Delta^2 W\left(\beta\frac{d\Delta}{dn} - \frac{dr}{dn}\right) + \Delta^2 WrD + \frac{2\Delta^3\beta'}{\rho} \\ &= -\Delta^3 h\beta' + \Delta^3 W\frac{d\beta}{dn} + \Delta^3\beta WD + \frac{2\Delta^3\beta'}{\rho} \\ &= \Delta^3\left[\left(\frac{2}{\rho}-h\right)\beta' + W\left\{\frac{d\beta}{dn}+\beta D\right\}\right]. \end{aligned} \tag{o}$$

$$\therefore \frac{1}{\Delta^3}\begin{vmatrix} E & F & G \\ L & M & N \\ A & B & C \end{vmatrix} = \left(\frac{2}{\rho}-h\right)\beta' + W\left(\frac{d\beta}{dn}+\beta D\right). \tag{o'}$$

Many other examples could be given, but enough has been shown to illustrate the use of vector methods in discussing certain forms found in differential geometry. The reader will see how much more direct and simple these expressions and reductions are in vector forms than in the use of the more tedious expressions in Cartesian coordinates.

UNIVERSITY OF TEXAS.

*On Certain Saltus Equations.**

By Henry Blumberg.

Introduction.

Instead of attempting to introduce into our discussion the utmost generality, we shall, for the sake of greater simplicity, confine ourselves to the consideration of a real, one-valued function $f(x)$, defined in the linear continuum, bounded at every point—and hence, according to the Borel theorem on sets of intervals, in every interval—and unrestricted as to continuity. From $f(x)$ we derive three new functions: $u(f, x)$, the upper-bound (=maximum) function; $l(f, x)$, the lower-bound (=minimum) function; and $s(f, x)=u(f, x)-l(f, x)$, the saltus (=oscillation) function of f.† Just as the notion of derivative at once suggests that of differential equation, so the notion of saltus leads to that of "saltus equation." It is the principal object of the present paper to give what may be regarded as complete solutions of such saltus equations in several simple cases. For the sake of greater brevity, we write as follows the successive saltus functions derived from $f(x)$:

$$s(f, x)=s_f'(x), \quad s(s_f', x)=s_f''(x), \quad s(s_f'', x)=s_f'''(x), \ldots.$$

Because of a theorem due to Sierpiński,‡ which asserts that $s_f'''(x)=s_f''(x)$ no matter what function $f(x)$ we start with, there is no need of considering saltus equations beyond the "second order," *i. e.*, equations involving a saltus with an index greater than 2. The saltus equations $s_f'(x)=g(x)f(x)$ and $s_f''(x)=g(x)s_f'(x)$, where $g(x)$ is an arbitrarily given continuous function and $f(x)$ is sought, are among those above referred to as "completely solved" in this paper.

The first section deals with several properties of functions of interest in themselves and useful later. Section 2 deals with the equation $s_f'(x)=g(x)f(x)$; Section 3, with the equation $s_f''(x)=g(x)s_f'(x)$; and the appendix indicates several lines of generalization.

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* Read before the American Mathematical Society, December 26, 1913.

† For the definitions of these functions see the author's paper, "On Certain General Properties of Functions," *Annals of Mathematics*, Vol. XVIII (1917), p. 147, and Hobson, "The Theory of Functions of a Real Variable" (1907), Art. 180.

‡ *Bulletin de l'Académie des Sciences de Cracovie* (1910), pp. 633–634.

Section 1. *Preliminary Theorems and Several Simple Saltus Equations.*

THEOREM I. *If two given functions are upper-semi-continuous (lower-semi-continuous) at a given point ξ, the saltus of their sum at ξ is greater than or equal to the saltus of each of the given functions at ξ.*

For let $f_1(x)$ and $f_2(x)$ be upper-semi-continuous at ξ. This condition is equivalent to the relations $f_1(\xi)=u(f_1,\xi)$ and $f_2(\xi)=u(f_2,\xi)$, where $u(f_1,x)$ and $u(f_2,x)$ are the upper-bound functions of f_1 and f_2, respectively. If $s'_{f_1}(\xi)=h$, a sequence $\{\xi_n\}$ of real numbers exists such that $\lim_{n\to\infty}\xi_n=\xi$ and $\lim_{n\to\infty} f_1(\xi_n)=u(f_1,\xi)-h=f_1(\xi)-h$. Moreover, since $f_2(\xi)$ is upper-semi-continuous, we have $\lim_{n\to\infty} f_2(\xi_n)\leqq f_2(\xi)$. Hence

$$\lim_{n\to\infty}[f_1(\xi_n)+f_2(\xi_n)]\leqq f_1(\xi)+f_2(\xi)-h,$$

which shows that the function $f(x)=f_1(x)+f_2(x)$ has a saltus $\geqq h$ at ξ. In the same way, we prove the theorem for $f_2(x)$, and for two lower-semi-continuous functions.

Since the saltus function $s'_f(x)=u(f,x)-l(f,x)$ is actually the sum of two upper-semi-continuous functions, $u(f,x)$ and $-l(f,x)$, we have, as an application of Theorem I,

THEOREM II. *The second saltus function is at every point greater than or equal to the saltus of both the upper-bound function and the lower-bound function.*

If in Theorem I it happens that the saltus of the sum is zero at ξ, then it follows that the saltus of both f_1 and f_2 is zero at ξ. That is, we have

THEOREM III. *If the sum of two given upper-semi-continuous functions is continuous at a point ξ, then each of the given functions is continuous as ξ.**

Either from Theorem II or Theorem III, we obtain by specialization

THEOREM IV. *If $s'_f(x)$ is continuous at a given point ξ, both $u(f,x)$ and $l(f,x)$ are continuous at ξ.*†

By means of Theorem IV we can obtain the complete solution of the saltus equation

$$s''_f(x)=0.$$

For from this equation it follows that $s'_f(x)$ is continuous, and hence, according to Theorem IV, both $u(f,x)$ and $l(f,x)$ are continuous functions. Conversely,

*The theorem is false for the *product* of two upper-semi-continuous functions.

†Although this theorem is near at hand and of sufficient interest to deserve mention among the general properties of functions, it does not seem to have been previously noted.

if $f(x)$ is such that $u(f,x)$ and $l(f,x)$ are continuous, it follows that $s_f'(x)$ is continuous, and hence $s_f''(x)=0$. Therefore, if $f_1(x)$ and $f_2(x)$ are any two continuous functions whatever, and we take the function $f(x)$ such that the "curve" whose equation is $y=f(x)$ lies everywhere between the curves $y=f_1(x)$ and $y=f_2(x)$ and has every point of each of these curves as a limiting point, then $f(x)$ is a solution of the saltus equation $s_f''(x)=0$; and every solution is so obtainable.

DEFINITION. The function $f(x)$ is said to be "*continuously bounded*" in a given interval (linear continuum) if $u(f,x)$ and $l(f,x)$ are continuous in the interval (linear continuum).*

We now have

THEOREM V. *A necessary and sufficient condition for a solution of the saltus equation $s_f''(x)=0$ is that it shall be continuously bounded.*

REMARK. *The saltus equation $s_f''(x)=g(x)$, where $g(x)$ is an arbitrarily given continuous function of x, is only apparently less restrictive upon $f(x)$ than the equation $s_f''(x)=0$ of Theorem V.* For since $s_f''(x)$ is zero in an everywhere dense set,† its continuity implies that it is identically zero.

DEFINITION. A function is said to be "*pointwise discontinuous at a given point ξ*," if it possesses at least one point of continuity in every neighborhood of ξ.‡

THEOREM VI. *The relation*

$$s_f''(\xi)=s_f'(\xi)$$

is a necessary and sufficient condition for the pointwise discontinuity of f at ξ.

For on the one hand, let $f(x)$ be pointwise discontinuous at ξ. Then every neighborhood of ξ contains at least one point where f is continuous, *i. e.*, where $s_f'(x)=0$. Therefore $l(s_f',\xi)=0$. Since $s_f'(x)$ is upper-semi-continuous, we have $u(s_f',\xi)=s_f'(\xi)$. Consequently

$$s_f''(\xi)=u(s_f',\xi)-l(s_f',\xi)=s_f'(\xi).$$

On the other hand, suppose $f(x)$ is not pointwise discontinuous at ξ. Then a neighborhood (α,β) of ξ exists such that f is totally discontinuous in (α,β), hence $s_f'(x)>0$ for $\alpha\leqq x\leqq\beta$. It follows, since an upper-semi-continuous function and hence $s_f'(x)$ attains its (greatest) lower bound in a closed interval,

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* Virtually all the definitions and theorems of this paper apply equally well to an interval and to the entire linear continuum. Further specific reference to this fact is regarded as unnecessary.

† *Annals of Mathematics, loc. cit.*, p. 151.

‡ Continuity at ξ implies pointwise discontinuity at ξ. We use "pointwise discontinuous" in the sense of "at most pointwise discontinuous," thus deviating somewhat from the more common usage for the sake of greater convenience.

that this lower bound of $s_f'(x)$ for $\alpha \leqq x \leqq \beta$ is a positive number, say h. Since, on account of the upper-semi-continuity of $s_f'(x)$, the relation $u(s_f', \xi) = s_f'(\xi)$ holds, we have

$$s_f''(\xi) = u(s_f', \xi) - l(s_f', \xi) = s_f'(\xi) - l(s_f', \xi) < s_f'(\xi) - h < s_f'(\xi).$$

Thus $s_f''(\xi) \neq s_f'(\xi)$ is a consequence of the fact that f is not pointwise discontinuous at ξ, and Theorem VI is completely proved.

From Theorem VI we obtain

THEOREM VII. *The set of solutions of the saltus equation $s_f''(x) = s_f'(x)$ is identical with the set of pointwise discontinuous functions.*

THEOREM VIII. *A necessary and sufficient condition that $f(x)$ be a solution of the saltus equation $s_f'(x) = f(x)$ is that it possesses the following two properties:* (α) f is upper-semi-continuous; (β) $f=0$ in an everywhere dense set.*

In the first place, the two properties are necessary. For (α) follows from the upper-semi-continuity of $s_f'(x)$. Since (α) implies that $u(f, x) = f(x)$, we have $s_f'(x) = u(f, x) - l(f, x) = f(x) - l(f, x)$ and therefore we must have $l(f, x) = 0$ to secure the relation $s_f'(x) = f(x)$. But $f(x)$, being upper-semi-continuous, attains its (greatest) lower bound, that is 0, in every closed interval; and hence (β) holds. In the second place, the properties are sufficient. For from (α), we conclude that $u(f, x) = f(x)$, and from (β) and (α), that $l(f, x) = 0$, whence $s_f'(x) = f(x)$.

Section 2. The Saltus Equation $s_f'(x) = g(x)f(x)$, where $g(x)$ is a Continuous Function.

(a) *If $g(\xi) \neq 0, 1$ we have $f(\xi) = 0$.* For let (α, β) be an interval containing ξ in its interior and such that $g(x) \neq 0, 1$ for $\alpha \leqq x \leqq \beta$. We may then write $f(x) = s_f'(x)/g(x)$, $\alpha \leqq x \leqq \beta$, which shows, in virtue of the upper-semi-continuity of $s_f'(x)$ that $f(x)$ is upper-semi-continuous in (α, β). $s_f'(x)$ being zero at every point of continuity of $f(x)$, must therefore vanish in an everywhere dense set S of (α, β). Consequently f vanishes in S, and therefore in virtue of Theorem VIII we have $s_f'(x) = f(x)$, $\alpha < x < \beta$. But the relation $s_f'(x) = f(x)$ would imply that $g(x) = 1$, contrary to our assumption, unless $f(x) = 0$. In particular, therefore, $f(\xi) = 0$.

(b) *If $g(\xi) = 1$, then $f(x)$ is upper-semi-continuous at ξ, and every neighborhood of ξ contains a zero of $f(x)$.* For consider the two possibilities: (1) ξ is an interior point of an interval (α, β) where $g(x)$ is constantly equal

* Cf. the author's paper, *Annals of Mathematics, loc. cit.*, p. 151.

to 1; (2) ξ is not in such an interval. If (1) holds, we have $s_f'(x)=f(x)$ for $\alpha<x<\beta$, and the assertions in question follow from Theorem VIII. If (2) holds, there are points where $g(x)\neq 0, 1$ in every neighborhood of ξ; hence, according to (a), every neighborhood of ξ contains zeros of $f(x)$. Consequently, since in virtue of our saltus equation a neighborhood of ξ exists in which $f(x)\geqq 0$, we have $l(f,\xi)=0$, whence $s_f'(\xi)=u(f,\xi)$. Therefore $u(f,\xi)=f(\xi)$, which is equivalent with the upper-semi-continuity of f at ξ.

(c) *If* $g(x)=0$, $f(x)$ *is continuous at* ξ.

(a), (b) and (c) give us necessary relations between the character of the given function $g(x)$ and a solution of our saltus equation. Furthermore, however, these relations completely characterize the solutions. For suppose $f(x)$ satisfies these relations. In the first place, let ξ be such that $g(\xi)\neq 0, 1$. Then ξ is contained in an interval throughout which $g(x)\neq 0, 1$. According to (a), $f(x)=0$ throughout this interval, so that $s_f'(\xi)=0$, $s_f'(\xi)=g(\xi)f(\xi)$. In the second place, let $g(\xi)=1$. Then, as an easy consequence of (a) and (b), we have $l(f,\xi)=0$. This relation, taken in conjunction with the upper-semi-continuity of f, shows that $s_f'(\xi)=f(\xi)=g(\xi)f(\xi)$. In the third place, let $g(\xi)=0$. Then f is continuous according to (c), and hence $s_f'(\xi)=0=g(\xi)f(\xi)$. The saltus equation is thus satisfied at every point, and we have

THEOREM IX. *If* $g(x)$ *is a given continuous function, then the set of solutions of the saltus equation* $s_f'(x)=g(x)f(x)$ *is identical with the set of functions* f *such that* (a) $f(x)=0$ *where* $g(x)\neq 0, 1$; (b) $f(x)$ *is upper-semi-continuous where* $g(x)=1$ *and* x *does not lie in the interior of an interval throughout which* $g(x)=1$; (c) $f(x)$ *is upper-semi-continuous and possesses an everywhere dense set of zeros in the interior of every interval throughout which* $g(x)=1$; *and* (d) $f(x)$ *is continuous where* $g(x)=0$.

The following implications of Theorem IX deserve mention.

THEOREM X. *The only solution of the saltus equation* $s_f'(x)=g(x)f(x)$, *where* $g(x)$ *is a given continuous function of* x *taking nowhere the values* 0, 1, *is* $f(x)\equiv 0$.

THEOREM XI. *The only solution of the saltus equation* $s_f'(x)=g(x)f(x)$, *where* $g(x)$ *is a given continuous function taking at no point the value* 1, *and throughout no interval the value* 0, *is* $f(x)\equiv 0$.

THEOREM XII. *A necessary and sufficient condition that* $f(x)$ *be a solution of the saltus equation* $s_f'(x)=g(x)f(x)$, *where* $g(x)$ *is a continuous function taking nowhere the value* 1, *is that* $f(x)$ *be continuous, and* $=0$ *where* $g(x)\neq 0$.

THEOREM XIII. *If no interval exists throughout which the continuous function $g(x)$ is 0 or 1, then all the solutions of the saltus equation $s_f'(x) = g(x)f(x)$ may be obtained by making $f(x)$ an arbitrary non-negative, upper-semi-continuous function in the set of points where $g(x)=1$ and giving it the value 0 elsewhere.**

Section 3. The Saltus Equation $s_f''(x)=g(x)s_f'(x)$, where $g(x)$ is a Continuous Function.

(a) *If $g(\xi)\neq 0, 1$, then $f(x)$ is continuous at ξ.* For an interval (α, β) with ξ as interior point exists, such that $g(x)\neq 0, 1$ for $\alpha \leqq x \leqq \beta$. Since $s_f''(x)$ is 0 in an everywhere dense set, it follows from our saltus equation that $s_f'(x)=0$ is an everywhere dense subset of (α, β). We now see that $s_f'(x)$ possesses in (α, β) the properties sufficient, according to Theorem VIII, to make it a solution of the saltus equation $s_F'(x)=F(x)$. Hence $s(s_f', x)=s_f'(x)$; *i. e.*, $s_f''(x)=s_f'(x)$ in (α, β). In particular, $s_f''(\xi)=s_f'(\xi)$, from which we would conclude, contrary to our assumption, that $g(\xi)=1$, unless $s_f'(\xi)=0$. Accordingly $f(x)$ is continuous at ξ.

(b) *If $g(\xi)=0$ and ξ is not an interior point of an interval where $g(x)$ is constantly 0, then $f(x)$ is continuous at ξ.* For since $s_f''(\xi)=0$, it follows that $s_f'(x)$ is continuous at ξ. As every neighborhood of ξ contains points where $g(x)\neq 0, 1$, we conclude from (a) that every neighborhood of ξ contains points where $s_f'(x)=0$. Hence $s_f'(\xi)=0$, and $f(x)$ is continuous at ξ.

(c) *If $g(x)=0$ throughout the interior of an interval (α, β), then $f(x)$ is continuously bounded for $\alpha<x<\beta$* (Theorem V).

(d) *If $g(\xi)=1$, then $f(x)$ is pointwise discontinuous at ξ* (Theorem VI).

Just as in the preceding section, the necessary conditions (a), (b), (c) and (d) upon a solution $f(x)$ of our present saltus equation are sufficient to characterize it completely. For suppose $f(x)$ satisfies these conditions. In the first place, let $g(\xi)\neq 0, 1$; then $f(x)$ is continuous at ξ and hence $s_f'(\xi)=0$. Since $s_f'(x)$ is upper-semi-continuous and non-negative, we have $s_f''(\xi)\leqq s_f'(\xi)$. Therefore $s_f''(\xi)=0$ and the equation $s_f''(x)=g(x)s_f'(x)$ is satisfied for $x=\xi$. In the second place, let $g(\xi)=0$ and let furthermore ξ be such that it is not an interior point of an interval where $g(x)$ is constantly 0; then $f(x)$ is continuous at ξ. Hence $s_f'(\xi)=0$, and therefore $s_f''(\xi)=0$, and $s_f''(\xi)=g(\xi)s_f'(\xi)$. That (c) and (d) are sufficient conditions follows from Theorems V and VI, respectively. We thus have

* The notion here employed of an upper-semi-continuous function defined in an arbitrary linear point-set requires no further explanation.

THEOREM XIV. *The set of solutions of the saltus equation* $s_f''(x) = g(x)s_f'(x)$, *where* $g(x)$ *is a given continuous function, is identical with the set of functions that are pointwise discontinuous at every point where* $g(x)=1$, *continuously bounded in the interior of every interval where* $g(x)$ *is constantly* 0 *and elsewhere continuous.**

A number of consequences of Theorem XIV deserve special mention:

THEOREM XV. *If the continuous function* $g(x)$ *is* 1 *nowhere and* 0 *throughout no interval, then the saltus equation* $s_f''(x)=g(x)s_f'(x)$ *has only the trivial solution* $f(x)=$*continuous function.*

THEOREM XVI. *If the continuous function* $g(x)$ *is nowhere* $=1$, *the set of solutions of the saltus equation* $s_f''(x)=g(x)s_f'(x)$ *is identical with the set of continuously bounded functions that are continuous where* $g(x)\neq 0$.

THEOREM XVII. *If there is no interval throughout which the continuous function* $g(x)$ *is* 0 *or* 1, *then all the solutions of the saltus equations* $s_f''(x) = g(x)s_f'(x)$ *are obtained by assigning arbitrary values to* $f(x)$ *at the points where* $g(x)=1$ *and making* $f(x)$ *continuous elsewhere.*

Appendix.

A. The coefficients in the saltus equations considered in this paper have so far been exclusively continuous functions. Without attempting a comprehensive treatment for the case of discontinuous coefficients, we shall now give a theorem relating to the saltus equation $s_f''(x)=g(x)s_f'(x)$ for a discontinuous g.

THEOREM XVIII. *If the zeros of* $g(x)$ *form (at most) a non-residual set*† *in every interval, then the set of solutions of the saltus equation* $s_f''(x)=g(x)s_f'(x)$ *is identical with the set of pointwise discontinuous functions that are continuous where* $g(x)\neq 1$.

For since $s_f'(x)$ is upper-semi-continuous, it is continuous in a residual set,‡ hence $s_f''(x)=0$ in a residual set. Therefore, according to the equation $s_f''(x)=g(x)s_f'(x)$, the product $g(x)s_f'(x)$ is equal to zero in a residual set. By hypothesis, the zeros of $g(x)$ constitute a non-residual set in every interval;

* A part of the content of Theorem XIV may be derived as follows: Taking the saltus of each side of the equation $s''_f(x)=g(x)s'_f(x)$, we obtain the equation $s'''_f(x)=g(x)s''_f(x)$. But according to Sierpiński's theorem, $s'''_f(x)\equiv s''_f(x)$. Therefore either $s''_f(x)=0$ or $g(x)=1$. In similar fashion, we may deal with the saltus equation of Section 2.

† According to Denjoy, an "*exhaustible*" set (= set of first category) is the sum of a denumerable set of non-dense sets; and a "*residual*" set, the complementary set of an exhaustible set. See *Journal de Mathématiques*, Sér. 7, Vol. I (1915), pp. 122–125.

‡ See, for example, Hobson, "The Theory of Functions of a Real Variable" (1907), p. 240.

consequently, the zeros of $s_f'(x)$ constitute a non-exhaustible set in every interval. The zeros of $s_f'(x)$ are therefore everywhere dense. Since additionally $s_f'(x)$ is upper-semi-continuous, we conclude from Theorem VIII that $s_f''(x)=s(s_f', x)=s_f'(x)$, which shows according to Theorem VII that f must be pointwise discontinuous. Furthermore, we now have two possibilities: (1) Either $s_f'(x)=0$, and then f is continuous at x. Or (2) $s_f'(x)\neq 0$, and then $g(x)=s_f''(x)/s_f'(x)=1$.

We have thus shown that f must necessarily be pointwise discontinuous, and continuous where $g(x)\neq 1$. But these conditions are also sufficient. For from the pointwise discontinuity of f, it follows that $s_f''(x)=s_f'(x)$, and hence our saltus equation is satisfied where $g(x)=1$; and from the continuity of f at other points, it follows that the saltus equation is also satisfied elsewhere.

One of the consequences of Theorem XVIII is

THEOREM XIX. *The only solutions of the saltus equation* $s_f''(x)=g(x)s_f'(x)$, *where* $g(x)$ *is a given function taking the value* 1 *nowhere and the value* 0 *in a non-residual set in every interval, are continuous functions.*

B. Although we have assumed throughout the paper that $f(x)$ is single-valued and bounded at every point, it may be readily seen that the dropping of the former assumption would necessitate hardly any change in our presentation, while the dropping of the latter would require only slight modification of the results and proofs. The extension of the treatment to functions of more than one variable is not difficult, but we shall not enter upon it here. We shall also not deal with the corresponding saltus equations that arise when the f-saltus, the e-saltus, the z-saltus, etc.,* are employed instead of the ordinary saltus.

C. In conclusion we call attention to the simple saltus equation $s_f'(x)=g(x)$, where $g(x)$ is an arbitrary discontinuous function. We have not succeeded in obtaining a satisfactory characterization of its solutions.

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* *Annals of Mathematics, loc. cit.*, pp. 148 and 149.

Investigations on the Plane Quartic.

By Teresa Cohen.

§ 1. *Introduction.*

The plane quartic $(\alpha x)^4$ is taken in the form

$$ax_0^4+4a_1x_0^3x_1+4a_2x_0^3x_2+6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3$$
$$+12mx_0x_1^2x_2+12nx_0x_1x_2^2+4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4.$$

It will be convenient first to mention briefly certain well-known forms connected with it that will be made use of in this article.

Of these several arise from the polar forms, $(\alpha x)^3(\alpha y)$, $(\alpha x)^2(\alpha y)^2$, $(\alpha x)(\alpha y)^3$. Since of each of these there is an ∞^2, the placing of one, two, or three conditions on the curves they represent results, respectively, in a locus for the pole, in a set of points, and in an invariant condition to be satisfied by the quartic.

Upon the polar line the only condition that can be imposed is its identical vanishing. This, as is well known, means that the quartic has a double point and requires the vanishing of the discriminant, an A^{27}.*

The polar conic may be made to break up into two lines. The locus of poles of such degenerate conics is the Hessian H, an A^3x^6. To make these two lines coincide requires two additional conditions and gives for the quartic an invariant, shown by Dr. Thomsen† to be an A^{48}.

From the polar cubic more can be obtained. The locus of poles of polar cubics having a double point is the Steinerian Σ, an $A^{12}x^{12}$. The cubic has also the invariants S and T, of degrees 4 and 6, respectively, which give rise to the covariants of the quartic $S\equiv A^4x^4$ and $T\equiv A^6x^6$. From the relation connecting the discriminant of the cubic with these two invariants we have

$$\Sigma\equiv 64S^3+T^2,$$

a very useful form for calculating coefficients of the Steinerian whenever that may be necessary. To require that the polar cubic have a cusp is two conditions, giving rise to a set of twenty-four points with cuspidal polar cubics. The number of these points is determined by the fact that they are common

* The notation $A^ix^j\xi^k$ is used to represent a comitant form of degree i in the coefficients of the quartic, j in x, and k in ξ.

† American Journal of Mathematics, Vol. XXXVIII (1916), p. 249.

points of S and T; therefore they are cusps of the Steinerian. There are also twenty-one points whose polar cubics have two double points and therefore break up into a line and a conic; these are the double points of the Steinerian. The line parts of these cubics will hereafter be referred to simply as "the twenty-one lines." Each of these lines meets the quartic so that the four tangents at the intersections are all on a point, which point is the corresponding point; *i. e.*, the pole of the cubic of which the line forms a part. If one of these lines be taken as $x_0=0$ and the corresponding point as $(1, 0, 0)$ then in $(ax)^4$

$$b_0=m=n=c_0=0.$$

The quartic also has certain contravariants obtained by imposing a condition on the four points in which a line cuts it. The locus of lines cutting the quartic in a self-apolar set is $s\equiv(s\xi)^4\equiv A^2\xi^4$. The locus of lines cutting in harmonic pairs is $t\equiv(t\xi)^6\equiv A^3\xi^6$.

§ 2. *The Undulation.*

If a line cuts the quartic in four consecutive points, the quartic is said to have an undulation. The invariant vanishing in this case is given by Salmon as an A^{60}. The undulation tangent is evidently a line of both s and t, and it is that special case of the twenty-one lines occurring when one of the lines is on its corresponding point. This corresponding point is the undulation itself, which is therefore a double point of Σ. Suppose the undulation to be at $(0, 1, 0)$ with the tangent x_0. Then

$$b=b_2=f=c_1=0.$$

The undulation is also a point of the Hessian, for its polar conic is

$$hx_0^2+2b_0x_0x_1+2mx_0x_2=x_0(hx_0+2b_0x_1+2mx_2),$$

a pair of lines. Let the double point of this conic be taken as $(0, 0, 1)$, which therefore becomes the corresponding point on the Steinerian. Then

$$m=0.$$

The polar cubic of $(0, 1, 0)$ is

$$a_1x_0^3+3hx_0^2x_1+3lx_0^2x_2+3b_0x_0x_1^2+3nx_0x_2^2.$$

This is made up of x_0 and a conic; its two double points are $(0, \sqrt{n}, \sqrt{-b_0})$ and $(0, \sqrt{n}, -\sqrt{-b_0})$; they are the two points of the Hessian corresponding to the double point of the Steinerian and are harmonic to the undulation point and to the Steinerian point corresponding to the undulation considered as a Hessian point. The terms in H not containing x_0 are

$$x_1^4x_2^2\cdot -cb_0^2+x_1^2x_2^4\cdot -2cb_0n+x_2^6\cdot -cn^2=-cx_2^2(b_0x_1^2+nx_2^2)^2.$$

Therefore x_0 is a triple tangent to the Hessian.

§ 3. *The Discriminant of the Hessian.*

The Hessian and Steinerian, as is well known, are not independent curves, but are brought into one-to-one correspondence by the relation

$$(\alpha x)(\alpha y)^2 a_i = 0, \quad i = 0, 1, 2,$$

where x is a point of the Steinerian and y a point of the Hessian. This says that the polar cubic of x has a double point y and the polar conic of y a double point x. The joins of all such pairs x and y give rise to the Cayleyan, an $A^{12}\xi^{18}$. Since x and y can not come together unless

$$(\alpha x)^3 a_i = 0, \quad i = 0, 1, 2,$$

which is the condition that the quartic have a double point, there is a very convenient reference scheme for handling these three curves in any other case. Let $(0, 1, 0)$ be a point of the Hessian, $(0, 0, 1)$ the corresponding point on the Steinerian, so that x_0 is a line of the Cayleyan. Then

$$a_1^2 a_2 a_i = 0, \quad i = 0, 1, 2,$$

or

$$m = b_2 = f = 0.$$

Now let us see under what conditions the Hessian acquires a double point. It is known to have one when the quartic does. To discover other cases let us use the above reference scheme. Then

$$H \equiv x_1^5 x_0 \cdot 2n(bh - b_0^2) + x_1^5 x_2 \cdot 2c_1(bh - b_0^2) + \text{lower terms in } x_1.$$

Therefore $(0, 1, 0)$ is a double point if

$$bh - b_0^2 = 0.$$

But this says that the polar conic of $(0, 1, 0)$, which is $hx_0^2 + 2b_0 x_0 x_1 + bx_1^2$, shall be a line counted twice, and this means the vanishing of an A^{48}. Since this line is already on $(0, 0, 1)$, let it be taken as x_1, so that

$$b_0 = h = 0.$$

Then the polar cubic of any point $(k, 0, 1)$ on x_1, namely

$$\begin{aligned}(ak + a_2)x_0^3 + 3(a_1 k + l)x_0^2 x_1 + 3(a_2 k + g)x_0^2 x_2 + 6(lk + n)x_0 x_1 x_2 \\ + 3(gk + c_0)x_0 x_2^2 + 3(nk + c_1)x_1 x_2^2 + (c_0 k + c)x_2^3,\end{aligned}$$

has a double point at $(0, 1, 0)$; therefore the Steinerian must contain the line x_1. Two of these polar cubics have cusps, the poles of which may be made $(1, 0, 0)$ and $(0, 0, 1)$ by requiring that

$$l = n = 0.$$

The cuspidal tangent of each cubic is on the pole of the other, and the pair of them make up the tangents to the Hessian at its double point. Then

$$S \equiv -a_1^2c_1^2x_0^2x_2^2 + x_0^3x_1(-ba_1a_2c_0 + ba_1g^2) + \ldots + x_1x_2^3(-ba_2c_0c_1 + bc_1g^2) + \text{higher powers in } x_1,$$

and

$$T \equiv 8a_1^3c_1^3x_0^3x_2^3 + x_0^5x_1 \cdot 4ba_1^3c_0^2 + \ldots + x_1x_2^5 \cdot 4ba_2^2c_1^3 + \text{higher powers in } x_1.$$

Therefore S touches x_1 and T has x_1 as a flex line at (1, 0, 0) and (0, 0, 1). Also

$$\Sigma \equiv x_1[x_0^3x_2^3 \cdot 64ba_1^6c_0^2c_1^3 + \ldots + x_0^3x_2^3 \cdot 64ba_1^3a_2^2c_1^6 + x_1\{x_0^{10} \cdot 16b^2a_1^6c_0^4 + \ldots + x_2^{10} \cdot 16b^2a_2^4c_1^6\} + \text{higher powers in } x_1],$$

showing that x_1 divides out only once, but that (1, 0, 0) and (0, 0, 1) are more than ordinary singularities.

The Hessian also has a double point when

$$c_1 = n = 0.$$

The symmetry shows that (0, 0, 1) is also a double point, as can be verified from the coefficients. Then (0, 1, 0) as a Hessian point has (0, 0, 1) as the corresponding Steinerian point, and vice versa. The polar conics of (0, 1, 0) and (0, 0, 1) are, respectively,

$$hx_0^2 + 2b_0x_0x_1 + bx_1^2 \quad \text{and} \quad gx_0^2 + 2c_0x_0x_2 + cx_2^2.$$

If the harmonic conjugate of x_0 as to the first pair of lines is taken as x_1 and that as to the second as x_2, then

$$b_0 = c_0 = 0.$$

This shows that x_0 is one of the twenty-one lines. To find the degree of the invariant vanishing in this case is one of the objects of this investigation. With the above reference scheme

$$S \equiv x_1x_2^3 \cdot -bcgl + x_2^2\{x_0^2(cgh^2 - chl^2) + x_0x_1 \cdot -bca_1g + x_1^2(-bcgh - bcl^2)\} + \text{lower powers in } x_2,$$

while

$$T \equiv x_1x_2^5 \cdot 2bc^2l^3 + x_2^4\{x_0^2 \cdot -3c^2h^2l^2 + 2x_0x_1(-3bc^2a_2hl + 6bc^2a_1l^2) + x_1^2(b^2c^2a_2^2 + 4b^2cg^3 + 12bc^2hl^2)\} + \text{lower powers in } x_2.$$

Therefore these two curves touch at (0, 0, 1) along the line x_1, and similarly at (0, 1, 0) along the line x_2. Then

$$\Sigma \equiv x_1^2x_2^{10} \cdot 4b^2c^4l^6 + x_2^9\{x_0^2x_1 \cdot -12bc^4h^2l^5 + x_0x_1^2 \cdot 8bc^2l^3(-3bc^2a_2hl + 6bc^2a_1l^2) + x_1^3 \cdot 4bc^2l^3(b^2c^2a_2^2 - 12b^2cg^3 + 12bc^2hl^2)\} + \text{lower powers in } x_2.$$

Therefore Σ has a singularity at (0, 0, 1), and, symmetrically, at (0, 1, 0), which is something more than merely a cusp. It may be a tac-node.

These three cases make up the totality of ways in which the Hessian may acquire a double point. Therefore its discriminant, an A^{225}, must be made up simply of the three invariants attached to them.

§ 4. *The Discriminant of* $(s\xi)^4$.

The contravariant $(s\xi)^4$ is on the line x_0 if

$$b_2=f=c=0.$$

The point of contact is given by

$$3c_1m\xi_1+(-bc_0+b_0c_1)\xi_2=0.$$

Since (0, 0, 1) is taken as any one of the points in which x_0 cuts the quartic, this point of contact will be on $(\alpha x)^4$ if $m=0$ (if $c_1=0$, the quartic would have an undulation). But this says that (0, 0, 1) is a point of the Steinerian. Therefore the forty-eight intersections of $(\alpha x)^4$ and $(s\xi)^4$, which considered as a point curve is of order twelve, are the same as those of $(\alpha x)^4$ and Σ. This can be substantiated by finding the point equation of $(s\xi)^4$. The line equation of $(\alpha x)^4$ is s^3-27t^2. But s formed for $(s\xi)^4$ is $-12S+A^3(\alpha x)^4$, where A^3 is the invariant A given by Salmon, and t formed for $(s\xi)^4$ is

$$T+(\alpha x)^4\cdot(t\alpha')^3(t\alpha'')^3(\alpha' x)(\alpha'' x).$$

Therefore the line equation of $(s\xi)^4$ is

$$[-12S+A^3(\alpha x)^4]^3-27[T+(\alpha x)^4\cdot(t\alpha')^3(t\alpha'')^3(\alpha' x)(\alpha'' x)]^2$$
$$=-27\Sigma+(\alpha x)^4\cdot A^{11}x^3.$$

x_0 will become a double line of s only if

$$b_0=c_0=m=0,$$

all other conditions than this placing more than one restriction on the quartic. The points in which x_0 meets the quartic are (0, 0, 1), (0, k, 1), (0, ωk, 1), (0, $\omega^2 k$, 1), where $k^3=-\frac{4c_1}{b}$ and $\omega^2+\omega+1=0$. The tangents to the quartic at these four points are, respectively,

$$\begin{aligned} c_1x_1 &=0,\\ 3nkx_0-3c_1x_1+3c_1kx_2 &=0,\\ 3n\omega kx_0-3c_1x_1+3c_1\omega kx_2 &=0,\\ 3n\omega^2kx_0-3c_1x_1+3c_1\omega^2kx_2 &=0, \end{aligned}$$

and they evidently have the common point $(c_1, 0, -n)$. Let $n=0$, so that this common point becomes (1, 0, 0). Since among the coefficients equated to zero are b_2, m, f, then (0, 1, 0) is a point of the Hessian with (0, 0, 1) as its corresponding Steinerian point. Since the tangent to the Steinerian at any

point is the polar line as to the quartic of the corresponding Hessian point, the tangent to the Steinerian at (0, 0, 1) is x_1, which is the tangent to $(ax)^4$ at the same point. Then since no distinction was made among the four points in which x_0 cuts the quartic by assigning the particular coördinates (0, 0, 1) to one of them, when one of the twenty-one lines is a line of $(s\xi)^4$, the quartic and Steinerian touch four times on this line, the four tangents going through the corresponding point. Also since the tangent to the Hessian at (0, 1, 0) is x_2, the tangents at the four Hessian points corresponding to these Steinerian points are also on that corresponding point.

x_0 will also be a line of $(s\xi)^4$ if

$$b=b_2=f=0;$$

i. e., if x_0 is a stationary line of the quartic. But this leads to nothing new. Therefore $(s\xi)^4$ has a double line only when one of the twenty-one lines is on $(s\xi)^4$, and its discriminant, an A^{54}, expresses this condition.

§ 5. *The Discriminant of* $(t\xi)^6$.

The contravariant $(t\xi)^6$ is on the line x_0 if

$$b=b_2=f=0.$$

Then

$$(t\xi)^6=2\xi_0^5\xi_1\cdot 2b_0c_1^2+ \text{ lower powers in } \xi_0.$$

x_0 becomes a double line either if $b_0=0$, which says that the quartic has a double point at (0, 1, 0) with x_0 as a tangent there, or if $c_1=0$, which says that the quartic has an undulation.

$(t\xi)^6$ is also on x_0 when

$$b_2=c_1=f=0.$$

Then

$$(t\xi)^6=2\xi_0^5\xi_1\cdot -bcn+2\xi_0^5\xi_2\cdot - bcm+ \text{ lower powers in } \xi_0.$$

For x_0 to be a double line requires either that $b=0$ or $c=0$, which repeats the undulation condition, or that $m=n=0$, which is the condition on the quartic under which the Hessian had two double points and for which the invariant of unknown degree vanishes. Then one of the twenty-one lines is a line of $(t\xi)^6$.

$(t\xi)^6$ also has x_0 as a line when $b=f=c=0$, but this leads to nothing new. Therefore the discriminant of $(t\xi)^6$, an A^{225}, must be made up of the discriminant of the quartic, an A^{27}, the undulation condition, an A^{60}, and the unknown invariant.

§ 6. *The Twenty-One Lines.*

The twenty-one lines * are given by an $A^i x^{21}$, where i is as yet unknown. That it can be determined is due to the fact that these lines are part of the common lines of two curves, all of whose common lines are known. These curves are the Cayleyan, an $A^{12}\xi^{18}$, and an $A^{15}\xi^{24}$, which is the locus of lines cutting the quartic so that three of the tangents at the intersections are on a point.† The common lines of these curves are are:‡ (1) the twenty-one lines counted sixteen times, since they are quadruple lines of both curves; (2) the twenty-four stationary lines, given by an $A^{24}x^{24}$, counted twice because they are double lines of $A^{15}\xi^{24}$; (3) the forty-eight lines of $(s\xi)^4$ at its intersections with the quartic. Since the polar points of these lines are on $(\alpha x)^4$, they are lines of the curve $(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)$, an $A^9\xi^{12}$, and are obtained by the elimination of ξ from $A^9\xi^{12}$, $(s\xi)^4 \equiv A^2\xi^4$, and (ξx) as an $A^{60}x^{48}$. The eliminant of $A^{12}\xi^{18}$, $A^{15}\xi^{24}$, and (ξx) is an $A^{558}x^{432}$, which must be made up of the common lines together with such invariants as express the fact that they become indeterminate. The only such invariant is the discriminant of the quartic, for if the quartic has a double point it divides out at least once from each of these curves.

$$A^{558}x^{432} = (A^i x^{21})^{16} \cdot (A^{24}x^{24})^2 \cdot A^{60}x^{48} \cdot (A^{27})^k.$$

$$\therefore\ 558 = 16i + 108 + 27k, \quad 16i + 27k = 450.$$

This equation shows that i must be a multiple of 9, and it can be satisfied only by $i=18$. Therefore the twenty-one lines are given by an $A^{18}x^{21}$.

§ 7. *The Invariant of Unknown Degree.*

The degree of the form giving the twenty-one lines being known, we can force one of them to lie on either $(s\xi)^4$ or $(t\xi)^6$. The condition so obtained will contain the undulation condition, however, for the tangent at an undulation is a special case of one of the twenty-one lines, and is a line of both s and t. Suppose the $A^{18}x^{21}$ factored into its component lines, the coördinates of each of

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* A curve on these lines is

$$A^9\xi^{12} \equiv 2(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha) - 3(s\xi)^4(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^2(s'''\alpha)^2 + \tfrac{1}{3}(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\alpha)^4.$$

† AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIX (1917), p. 227.

‡ Let $b_2 = m = f = 0$, so that x_0 is a line of the Cayleyan. Let $n=0$, so that x_2 is the tangent to the Hessian at (0, 1, 0). The tangent to the Steinerian can not be chosen as the other reference line because the stationary lines are known to be among the common lines of the two curves and the Steinerian is tangent to them. Instead, let (1, 0, 0) be determined by the intersection of x_2 with the polar line of (0, 0, 1). Then $c_0 = 0$. The coefficient of ξ_0^{24} in $A^{15}\xi^{24}$ becomes $-\frac{1}{3}^0 b^2 c b_0^4 c_1^8$. If $b=0$, x_0 is a stationary line of the quartic. If $c=0$, the quartic and Σ meet at (0, 0, 1) and x_0 is a line of $(s\xi)^4$. If $b_0=0$, x_0 is one of the twenty-one lines. If $c_1=0$, an invariant condition is imposed on the quartic.

these substituted in $(s\xi)^4$, the twenty-one resulting terms multiplied together, and the coefficients of the $A^{18}x^{21}$ substituted for symmetric functions of these coördinates. The result is an

$$A^{18\cdot 4+2\cdot 21}=A^{114}=A^{60}\cdot A^{54}.$$

This leads to nothing new. But the same process applied to $(t\xi)^6$ gives the degree of the required invariant. For we obtain $A^{18\cdot 6+21\cdot 3}=A^{171}$ from which the undulation condition must divide out at least once. Therefore the invariant in question may be either of degree 111 or 51. The first of these will not fit into the discriminant of the Hessian, but the second would give

$$A^{225}=A^{27}\cdot (A^{48})^2\cdot (A^{51})^2,$$

while for the discriminant of $(t\xi)^6$ we should have

$$A^{225}=(A^{27})^2\cdot (A^{60})^2\cdot A^{51}.$$

Therefore we have an invariant of degree 51, the vanishing of which expresses the fact that the Hessian has two double points, that $(t\xi)^6$ has a double line, and that one of the twenty-one lines is a line of $(t\xi)^6$. It also has another meaning, as will appear later.

§ 8. *Certain other Invariants.*

The degree of the form giving the twenty-one lines being known, that for the twenty-one corresponding points can be calculated from the fact that each point is the polar point of the corresponding line as to both $(s\xi)^4$ and $(t\xi)^6$. Suppose, as before, the $A^{18}x^{21}$ factored into its component lines, the coördinates substituted for η in $(s\xi)\,(s\eta)^3$, and the symmetric functions of these coördinates in the product of the resulting terms replaced by the coefficients of $A^{18}x^{21}$. The result gives the twenty-one points together with such invariants as express that the polar point becomes indeterminate (*i. e.*, that one of the lines be a double line of s). Then

$$A^{18\cdot 3+2\cdot 21}\xi^{21}=A^{54}\cdot A^{42}\xi^{21}.$$

Similarly from $(t\xi)^6$ we obtain

$$A^{18\cdot 5+3\cdot 21}\xi^{21}=A^{60}\cdot A^{51}\cdot A^{42}\xi^{21}.$$

Therefore the twenty-one points are given by an $A^{42}\xi^{21}$.* Suppose one of them to be on $(\alpha x)^4$, and therefore on its own polar cubic. Then, as before, considering $A^{42}\xi^{21}$ broken up into its component points, the coördinates substituted

* This degree fits in with the form $A^{36}\xi^{18}$ for the line equation of Σ. For the cusps are obtained from the eliminant of S, T, and (ξx) as an $A^{45}\xi^{24}$. Then from the Plücker formula

$$A^{12\cdot 22}\xi^{12\cdot 11}=A^{264}A^{132}=(A^{42}\xi^{21})^2\cdot (A^{45}\xi^{24})^3\cdot A^{36}\xi^{18}.$$

in $(\alpha x)^4$, and their symmetric functions replaced in the product of the resulting terms, we obtain an $A^{42\cdot 4+21}=A^{189}$. Out of this the condition for an undulation, when the point is on the line part of its polar cubic, must divide out at least once. If it is no more than once, we have left an A^{129} to express that the point is on the conic part of its polar cubic. To substantiate this degree we must find some case into which the undulation condition does not enter. We had before that the point is (1, 0, 0) and its corresponding line x_0 if $b_0=m=n=c_0=0$. To put the point on the quartic requires that $a=0$. The undulation case is excluded because our choice of coördinates for the point and line implies that they are not incident. We can still choose (0, 1, 0) and (0, 0, 1) on x_0. Let us take them as the double points of the polar cubic of (1, 0, 0), so that $g=h=0$. Then in $(s\xi)^4$ the coefficients of ξ_1^4, ξ_2^4, $\xi_1^3\xi_2$, $\xi_1\xi_2^3$ vanish, showing that x_1 and x_2 are lines of s with contact (1, 0, 0), so that (1, 0, 0) is a double point of $(s\xi)^4$.* It is known that the double lines of $(\alpha x)^4$ are given by an $A^{16}x^{28}$; therefore the double points of $(s\xi)^4$ are given by an $A^{32}\xi^{28}$. One of them can lie on $(\alpha x)^4$ only in our present case or when $(\alpha x)^4$ has a double point, for them $(s\xi)^4$ has that same double point with the same tangents. Then by the method of considering the $A^{32}\xi^{28}$ factored into its component parts we have as the condition that one of these points be on $(\alpha x)^4$ an

$$A^{32\cdot 4+28}=A^{156}=A^{129}\cdot A^{27}.$$

This establishes the degree of the A^{129}.

The totality of polar cubics of the twenty-one points given by the $A^{42}\xi^{21}$ is given by an $A^{63}x^{63}$. Suppose this factored into its components cubics $(\beta_i x)^3$, and then each of these factored into its component line and conic $(\gamma_i x)\cdot(\delta_i x)^2$. Then the condition that the line touch the conic is of second degree in both γ_i and δ_i, therefore of second degree in β_i. Therefore the invariant expressing the condition that there be a polar cubic made up of a conic and a line touching it is an $A^{63\cdot 2}=A^{126}$. This result can be substantiated in other ways. If, as usual, we take the line as x_0 and the corresponding point as (1, 0, 0) so that

$$b_0=m=n=c_0=0,$$

then the conic will touch x_0 at (0, 1, 0) if

$$h=l=0.$$

Then S goes through (1, 0, 0) while T has a double point there, so that Σ has two cusps and a double point coming together and acquires a triple point, since x_0^9 is the highest power of x_0 appearing in its equation. Since two of the

*An examination of the coefficients of $(s\xi)^4$ shows that an undulation is not a double point of it.

intersections of S and T have come together, the A^{126} must be a factor of the tact-invariant of these two curves. We have seen (§ 3) that in case of the vanishing of either the A^{48} or the A^{51} that S and T touch at two distinct points, and these three cases seem to be the only ones in which two intersections of these curves come together. Their tact-invariant is found from the formula given by Salmon to be of degree $4\cdot 6(6+8-3)+6\cdot 4(4+12-3)=576$. Then

$$A^{576}=(A^{48})^2\cdot(A^{51})^2\cdot(A^{126})^3.$$

Also, inspection of their coefficients shows that both H and S touch x_0 at (0, 1, 0). As will be seen later, two intersections of S and H can come together only when two of S and T do. When the A^{51} vanishes H has two double points on S; when the A^{48} vanishes, H has a double point which is also a double point of S. The tact-invariant of S and H is of degree $4\cdot 6(6+8-3)+3\cdot 4(4+12-3)=420$.

$$A^{420}=(A^{48})^4\cdot(A^{51})^2\cdot A^{126}.$$

The twenty-one polar cubics being given by an $A^{63}x^{63}$, while the twenty-one line parts are given by an $A^{18}x^{21}$, the twenty-one conic parts are given by an $A^{45}x^{42}$. Since the discriminant of a conic is of third degree in its coefficients, the condition that one of these conics break up is an A^{135}. But then the polar cubic consists of three lines, any two of which may be considered as the conic part, so that this degree is probably divided by three. Therefore the condition that there be a polar cubic made up of three lines is an A^{45}. Then three of the double points of the Steinerian have come together, giving it a triple point.*

§ 9. *The Eliminant of S, T, H.*

Under the reference scheme where

$$b_2=f=m=0,$$

the coefficient of the highest term in x_2 in both S and T is a power of (c_1l+n^2). Then both coefficients vanish if $c_1=n=0$, but this, as we have seen, means the

* Professor Morley has pointed out to me that the degrees of these last two invariants can be substantiated by the work of Caporali (Memorie di Geometria, p. 171) on a web of plane curves. If i curves of the web are of a particular kind, then the invariant expressing that there be a curve of that kind in a net is of degree i in the coefficients of the three curves upon which the net is built up. From his results the invariants expressing that in a net of cubics there be a cubic with three double points or a cubic with two coincident double points are of degree 15 and 42, respectively, in the coördinates of the three cubics. But in the polar net each cubic is linear in the coefficients of the quartic. Therefore the invariants are are of degree 45 and 126 in the coefficients of the quartic.

vanishing of an A^{51}. Therefore for the general quartic we obtain (0, 0, 1) as one of the common points of S and T if we make

$$l=n=0.$$

But then the coefficient of x_1^4 vanishes in S. Therefore to the twenty-four cusps of the Steinerian correspond those twenty-four points of the Hessian which make up its intersection with S.

It is known that the polar cubic of a point on S can be reduced to the sum of the cubes of three linear factors; its Hessian is the product of these factors and is one of those four triangles whose sides pass through all the flexes of the cubic. If the cubic has a double point, there is only one such proper triangle, made up of the tangents at the double point and a line through the three remaining flexes (flex line); in case of a cusp even this degenerates to the cusp tangent counted twice and the line joining the cusp to the sole remaining flex. Here the polar cubic of (0, 0, 1) is

$$a_2x_0^3+3gx_0^2x_2+3c_0x_0x_2^2+3c_1x_1x_2^2+cx_2^3,$$

and its Hessian is

$$-c_1^2x_2^2(a_2x_0+gx_2).$$

But x_2 is the tangent to H at (0, 1, 0) and $a_2x_0+gx_2$ is the tangent to S at the same point. Therefore the polar cubic of a cusp of the Steinerian has as its cusp tangent and its flex line the tangents to H and S, respectively, at the corresponding point. The cubic can be thrown into the form

$$\frac{1}{a_2^2}\left[(a_2x_0+gx_2)^3+x_2^2\{3a_2(a_2c_0-g^2)x_0+3a_2^2c_1x_1+(ca_2^2-g^3)x_2\}\right]$$

which also shows up the tangent at the flex as the linear form in brackets.

The polar cubic of (0, 1, 0) is

$$a_1x_0^3+3hx_0^2x_1+3b_0x_0x_1^2+bx_1^3+c_1x_2^3$$

and its Hessian is

$$c_1x_2[x_0^2(a_1b_0-h^2)+x_0x_1(ba_1-b_0h)+x_1^2(bh-b_0^2)]$$

Then the tangent to H at (0, 1, 0) passes through three flexes of the cubic, the three flex tangents pass through the corresponding Steinerian point and the binary Hessian of these three tangents picks up the six remaining flexes.

The coefficient of x_2^6 in H is $-c_1^2g$. Since we cannot have $c_1=0$ without bringing on the vanishing of the A^{51} and worse, let us make $g=0$. Then we have made S, T, H have a common point at (0, 0, 1). The polar cubic of (0, 0, 1) becomes

$$a_2x_0^3+x_2^2(3c_0x_0+3c_1x_1+cx_2).$$

If we let the intersection of stationary tangent and cusp tangent be (1, 0, 0), then

$$c_0=0.$$

But since we now have $g=n=c_0$, the Steinerian point corresponding to (0, 0, 1) as a Hessian point is (1, 0, 0), which also becomes a cusp on the Steinerian. So to sum up, we have: (1) the cusp e_0 on the Steinerian corresponds to e_2 on the Hessian; (2) the cusp e_2 on the Steinerian corresponds to e_1 on the Hessian; (3) the tangent to the Hessian at e_1 goes through e_0 and at e_2 goes through e_1; (4) the polar cubic of e_2 has the tangent to the Hessian at e_1 as cusp tangent, the line $x_0(=e_1e_2)$ as flex line, and a stationary tangent going through e_0; (5) the polar cubic of e_0 has an analogous set of lines; (6) S is tangent to x_0 at e_1 and to x_1 at e_2.

The real underlying significance of this state of affairs is not known; neither is the degree of the invariant representing it, but it can be guessed at. For the eliminant of S, T, H is an A^{360}. The only other condition under which these three curves have a common point is when the A^{51} vanishes, but then they have four common points, for H has two double points on the other curves. Then presumably the A^{51} divides four times out of the eliminant, leaving an A^{156}. The essential difference between the two cases seems to be this: To every ST point (intersection of S and T) there is attached an SH point with which it cannot coincide without more than one condition on the quartic. In order for S, H, T to have a common point an ST point must coincide with an SH point. If then the corresponding SH and ST points also coincide, the A^{51} vanishes. When there is no further coincidence, we have the other case.

§ 10. *The Eliminant of the Polar Cubic, Conic, and Line.*

Suppose we ask that the polar cubic, conic, and line of (0, 1, 0), which is not on the quartic, have a common point (0, 0, 1). Then

$$b_2=f=c_1=0.$$

The symmetry of this condition shows that the relation between (0, 1, 0) and (0, 0, 1) is a mutual one. Also x_0 is a line of $(t\xi)^6$. Therefore we might define $(t\xi)^6$ as the locus of the joins of pairs of points so related that the polar curves of each pass through the other, the pair of points being the common harmonic pair of the two harmonic pairs of points in which their join cuts the quartic.

In general the eliminant of the polar curves of x, which are an Ax^3y, an Ax^2y^2, and an Axy^3, is an $A^{11}x^{26}$. Out of this $(\alpha x)^4$ divides twice, leaving an

A^9x^{18}. Before asking the meaning of this latter curve, let us examine the eliminant of a cubic Ay^3, its polar conic Axy^2, and its polar line Ax^2y. It is an $A^{11}x^{15}$, from which the cubic divides out twice. The remainder, an A^9x^9, gives the nine stationary lines. Therefore the A^9x^{18} is for the quartic the locus of points lying on the stationary lines of their polar cubics, also the locus of flexes of polar cubics the tangent at which lies on the pole.

Salmon gives as the equation of the nine stationary lines of the cubic.

$$5SU^2H-H^3-U\theta,$$

where S is the invariant of degree 4, U is the cubic itself, H is its Hessian and θ is a certain A^8x^6. For a polar cubic of the quartic the first three pass over readily enough into the covariant S, the quartic itself, and its Hessian, but θ is not so easily transferred. So we will substitute for it another A^8x^6, expressible in terms of S, U, H, and θ, which gives the locus of a point whose polar conic as to the cubic is on its polar line as to the Hessian. Then it is easily shown that

$$A^9x^{18}=8SU^2H-H^3+9U\theta',$$

where S is the covariant S of the quartic, U the quartic itself, H its Hessian, and θ' the locus of those points whose polar conics as to the quartic are on their polar lines as to the Hessian.

The form of the A^9x^{18} shows that all of its intersections with $(\alpha x)^4$ are used up at the flexes of the quartic. The twenty-four points are also part of its intersection with H, but there are eighty-four more, lying on θ'. Now we saw that (0, 1, 0) is a point of the A^9x^{18} if

$$b_2=f=c_1=0.$$

To make it a point of H also requires that

$$m=0.*$$

But there are only forty-two such points, for they are the points where H, $(t\xi)^6$ and the Cayleyan all touch.† Then the intersections of H and θ' must come together by twos, and, indeed, it is readily shown that the two curves touch at their common points.

It is also easily shown that the forty-two inflexions of the Steinerian are at the points corresponding to these Hessian points.

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*Or symmetrically, (0, 0, 1) is a point of H if $n=0$.

†Proc. Nat. Ac. Sci., Vol. III (1917), p. 449.

§ 11. *The Polar Conic of Two Points.*

The eliminant of the polar curves of the quartic connects up with the polar conic of two points in the following way. The polar conic of x and y as to the quartic is $(ax)(ay)(az)^2$. Any point on the line xy may be represented parametrically as $x+\lambda y$. This line meets the conic in two points whose parameters λ are given by

$$(ax)(ay)(\overline{ax+\lambda y})^2=(ax)^3(ay)+2\lambda(ax)^2(ay)^2+\lambda(ax)(ay)^3=0.$$

If we ask that these points become indeterminate, we ask that the coefficients of λ all vanish. Then x and y are a pair of points such that the polar curves of each are on the other, so that we can define the A^9x^{18} of the last section as the locus of pairs of points whose polar conic is not only degenerate but contains their join, and $(t\xi)^6$ as the locus of this join. If x and y be taken as (0, 1, 0) and (0, 0, 1), then, as we have seen,

$$b_2=f=c_1=0.$$

The polar conic of the two points is

$$lx_0^2+2mx_0x_1+2nx_0x_2=x_0(lx_0+2mx_1+2nx_2).$$

To make the double point at one of the points requires that

$$m=0 \quad (\text{or } n=0).$$

But this makes (0,1,0) one of the forty-two points on H where it touches $(t\xi)^6$.

In general for a given y we have a whole locus of points x, the polo-Hessian of y to be exact, such that the polar conic of x and y is a pair of lines. We have seen that for one of these lines to be xy, x and y must lie on an A^9x^{18}. If we ask instead that the double point be at x, then x is a point of the Hessian and y a point of the Steinerian. For the polar conic of (0,1,0) and (0,0,1) is

$$lx_0^2+2mx_0x_1+2nx_0x_2+b_2x_1^2+2fx_1x_2+c_1x_2^2,$$

and this becomes two lines on (0, 1, 0) if

$$m=b_2=f=0,$$

being, in fact, the tangents at the double point of the polar cubic of the Steinerian point. To ask both that xy be one of the lines and that the double point be at x picks out, as we have seen, the forty-two inflexions of the Steinerian, and the forty-two points where the Hessian touches the Cayleyan and $(t\xi)^6$.

Making a fresh start, let us ask that the polar conic of (0,1,0) and (0,0,1) be the square of the line. If we take (1, 0, 0) as a point of this line, then

$$l=m=n=0, \quad b_2c_1-f^2=0.$$

But then we find that the coefficients of x_1^4 and x_2^4 in S vanish. Therefore the pair of points whose polar conic is a repeated line lie on S. But we already have a correspondence of the points of S, for the polo-Hessian of a point of S is made up of three lines, the intersections of which are on S. With the above reference scheme

$$S_{400} = (a_1b_0 - h^2)(-a_2c_0 + g^2).$$

(1, 0, 0) was any point of the repeated line. Let us take it as one of the intersections with S by making

$$a_1b_0 - h^2 = 0.$$

Then the polo-Hessian of (0, 1, 0) becomes

$$x_1[x_0x_1(ba_1f - a_1b_2^2 - b_0fh) + x_0x_2(ba_1c_1 - a_1b_2f - b_0c_1h) + x_1^2(bfh - b_0^2f - b_2^2h) + x_1x_2(bc_1h - b_0^2c_1 - b_2fh)].$$

The double points are (1, 0, 0), (0, 0, 1), and

$$y \equiv (bc_1h - b_0^2 - b_2fh,\ -ba_1c_1 + a_1b_2f + b_0c_1h,\ ba_1f - a_1b_2^2 - b_0fh).$$

Then the polar conic of (0, 1, 0) and (1, 0, 0) becomes $a_1x_0^2 + 2hx_0x_1 + b_0x_1^2$, that of (0, 1, 0) and (0, 0, 1) is $b_2x_1^2 + 2fx_1x_2 + c_1x_2^2$, and that of (0, 1, 0) and y is x_1^2. Therefore the polar conic of a point on S and any vertex of its polo-Hessian triangle is the square of the opposite side.*

If, now, we also ask that the repeated line be on (0, 1, 0) we have

$$b_2 = f = 0.$$

Then (0, 1, 0) is one of the twenty-four intersections of S and H. Therefore the polar conic of an SH point and its corresponding ST point is the square of a line on the SH point (the tangent to H there).

If, finally, we ask that the polar conic of (0, 1, 0) and (0, 0, 1) be x_0^2, we have

$$m = n = b_2 = f = c_1 = 0.$$

But then the A^{51} vanishes. Therefore we have a new definition for the A^{51}; it is the invariant expressing the condition that there be a pair of points such that their polar conic as to the quartic is the square of the line joining them.

§ 12. *Salmon's Connex.*

Salmon has shown that with any plane curve $(ax)^n$ there is associated an $A^3x^{2(n-2)}y^{n-2}$, which when x is a point on the curve picks out the remaining

* We might expect this sort of (3, 1) correspondence from the following considerations: The line equation of $(ax)(ay)(az)^2$ is an $A^2x^2y^2\xi^2$. If the conic is the square of a line, the line equation vanishes identically. Equating each coefficient to zero, we have six equations from which we can eliminate the six quantities y_0^2, y_0y_1, y_0y_2, y_1^2, y_1y_2, y_2^2, giving an $A^{12}x^{12}$. This must be S^2.

intersections of the tangent at x with the curve. This connex is expressible in terms of polars of the Hessian of the curve itself and the Hessians of the polar curves of x as to $(ay)^n$. In case of the quartic the connex is an $A^3x^4y^2 \equiv 15\,(hx)^4(hy)^2 - 9\,(h_1x)\,(h_1y)^2$, where $(hy)^6$ is the Hessian of the quartic and $(h_1y)^3$ is the Hessian of the polar cubic of x. Explicitly

$$
\begin{aligned}
A^3x^4y^2 = \overset{3}{\Sigma} x_0^4 [\, & y_0^2 \cdot 6\,(agh - al^2 - a_1^2 g - a_2^2 h + 2a_1a_2 l) \\
& + \overset{2}{\Sigma} y_0y_1 \cdot 4\,(ab_0 g + ahn - 2alm - a_2^2 b_0 - a_1^2 n + 2a_1a_2 m - a_1 gh + a_1 l^2) \\
& + \overset{2}{\Sigma} y_1^2 (abg - ba_2^2 + ab_0 n - 2ab_2 l + afh - am^2 + 2a_1a_2b_2 - a_1^2 f - a_1 b_0 g \\
& \quad - a_1 hn + 2a_1 lm) \\
& + y_1y_2 (ab_0c_0 + 2ab_2 g + 2ac_1 h - 4afl - amn - 2a_1^2 c_1 - 2a_2^2 b_2 \\
& \quad - 4a_1a_2 f - a_1 c_0 h - a_2 b_0 g - a_1 gm - a_2 hn + 2a_1 ln + 2a_2 lm)\,] \\
+ \overset{6}{\Sigma} x_0^3 x_1 [\, & y_0^2 \cdot 8\,(ab_0 g + ahn - 2alm - a_2^2 b_0 - a_1^2 n + 2a_1a_2 m - a_1 gh + a_1 l^2) \\
& + y_0y_1 \cdot 2\,(abg - ba_2^2 + 7ab_0 n - 2ab_2 l + afh - 7am^2 + 2a_1a_2b_2 - a_1^2 f \\
& \quad + 5a_1 b_0 g - 12a_2 b_0 l - 7a_1 hn + 12a_2 hm + 2a_1 lm - 6gh^2 + 6hl^2) \\
& + y_0y_2 \cdot (7ab_0c_0 + 2ab_2 g + 2ac_1 h - 4afl - 7amn - 2a_1^2 c_1 - 2a_2^2 b_2 \\
& \quad + 4a_1a_2 f - 7a_1 c_0 h - 7a_2 b_0 g + 17a_1 gm + 17a_2 hn - 10a_1 ln \\
& \quad - 10a_2 lm - 12ghl + 12l^3) \\
& + y_1^2 \cdot 3\,(abn + ab_0 f + ba_1 g - 2ab_2 m - 2ba_2 l + 2a_2b_2 h - a_1 fh - b_0 gh \\
& \quad - hn^2 + 2hlm) \\
& + y_1y_2 (abc_0 + 5ab_0c_1 - ba_2 g + 3ab_2 n - 9afm + 2a_1b_0c_0 + 7a_1b_2 g \\
& \quad - 5a_1c_1 h - 4a_2b_0 n - 10a_2b_2 l + 11a_2 fh - 3c_0h^2 - 2a_1 fl - 3b_0 gl \\
& \quad - 2a_1 mn + 4a_2 m^2 - 3ghm + 3hln + 6l^2 m) \\
& + y_2^2 (2acb_0 + ab_2c_0 - 2ca_1 h - 3ac_1 m - 2a_2b_0c_0 - a_2b_2 g + 5a_2c_1 h \\
& \quad + 4a_1c_0 m - 2a_1c_1 l + 4a_1 fg - 4a_2 fl - 3c_0 hl - 4a_1 n^2 + 2a_2 mn \\
& \quad - 3glm + 6l^2 n)\,] \\
+ \overset{3}{\Sigma} x_0^2 x_1^2 [\, & \overset{2}{\Sigma} y_0^2 \cdot 3\,(abg - ba_2^2 + 3ab_0 n - 2ab_2 l + afh - 3am^2 + 2a_1a_2b_2 - a_1^2 f \\
& \quad + a_1 b_0 g - 4a_2 b_0 l - 3a_1 hn + 4a_2 hm + 2a_1 lm - 2gh^2 + 2hl^2) \\
& + y_0y_1 \cdot 6\,(abn + ab_0 f + ba_1 g - 2ab_2 m - 2ba_2 l + 2a_2b_2 h + 4a_1 b_0 n \\
& \quad - a_1 fh - b_0 gh - 4a_1 m^2 - 4b_0 l^2 - 5h^2 n + 10hlm) \\
& + \overset{2}{\Sigma} y_0y_2 \cdot 3\,(abc_0 + ab_0c_1 - ba_2 g - ab_2 n - afm + 2a_1b_0c_0 + 3a_1b_2 g \\
& \quad - a_1c_1 h + 4a_2b_0 n - 2a_2b_2 l + 3a_2 fh - 3c_0h^2 - 2a_1 fl - 7b_0 gl \\
& \quad - 2a_1 mn - 4a_2 m^2 + 5ghm - hln + 6l^2 m) \\
& + y_2^2 (abc - ab_2c_1 - ba_2c_0 + 2ca_1b_0 - 3ch^2 + 4a_1b_2c_0 + 4a_2b_0c_1 - 6a_1c_1 m \\
& \quad - 6b_0c_0 l - 3a_2 fm - 3b_2 gl + 3c_0 hm + 3c_1 hl + 6fgh - 3fl^2 - 3gm^2 \\
& \quad - 6hn^2 + 12lmn)\,]
\end{aligned}
$$

$$+\sum^{3}x_0^2x_1x_2[y_0^2\cdot 3(3ab_0c_0+2ab_2g+2ac_1h-4afl-3amn-2a_1^2c_1-2a_2^2b_2$$
$$+4a_1a_2f-3a_1c_0h-3a_2b_0g+5a_1gm+5a_2hn-2a_1ln-2a_2lm$$
$$-4ghl+4l^3)$$
$$+\sum^{2}y_0y_1\cdot 3(abc_0+3ab_0c_1-ba_2g+ab_2n-5afm+6a_1b_0c_0+5a_1b_2g$$
$$-3a_1c_1h+4a_2b_0n-6a_2b_2l+7a_2fh-7c_0h^2-2a_1fl-13b_0gl$$
$$-6a_1mn-4a_2m^2+9ghm+hln+10l^2m)$$
$$+\sum^{2}y_1^2\cdot 3(abc_1+2ba_1c_0-ab_2f-3bgl+4a_2b_0f-2b_0c_0h-4a_2b_2m$$
$$+5b_2gh-c_1h^2-2a_1fm-2b_0gm-fhl+4lm^2)$$
$$+y_1y_2(abc+8ab_2c_1+2ba_2c_0+2ca_1b_0-9af^2-3bg^2-3ch^2+16a_1b_2c_0$$
$$+16a_2b_0c_1-12a_1c_1m-12a_2b_2n-6a_1fn-6a_2fm-18b_0gn$$
$$-12b_2gl-18c_0hm-12c_1hl+30fgh+6gm^2+6hn^2+24lmn)].$$

This connex may be considered either as a conic or a quartic, depending on whether x or y is the given point. If the curve has a double point the conic vanishes, but the quartic does not. Let

$$a=a_1=a_2=0,$$

so that (1, 0, 0) is a double point, and let this be taken as the point y. Then

$$A^3x^4y^2=\sum^{2}x_0^2x_1^2\cdot 3(-2gh^2+2hl^2)+x_0^2x_1x_2\cdot 3(-4ghl+4l^3)$$
$$+\sum^{2}x_0x_1^3\cdot 3(-b_0gh-h^2n+2hlm)$$
$$+\sum^{2}x_0x_1^2x_2\cdot 3(-c_0h^2-2b_0gl-ghm+4l^2m)$$
$$+\sum^{2}x_1^4(bgh-bl^2-b_0^2g-b_0hn+2b_0lm)$$
$$+\sum^{2}x_1^3x_2(-b_0c_0h+4b_2gh-4b_0gm+2b_0ln-4b_2l^2-3hmn+6lm^2)$$
$$+x_1^2x_2^2(-3b_0gn-3c_0hm+6fgh-6fl^2-3gm^2-3hn^2+12lmn).$$

Obviously this curve has a double point at (1, 0, 0) with the same tangents as the quartic. Therefore six of the sixteen intersections of the two curves are used up here. To find the others let us take (0, 1, 0) on $(ax)^4$, so that $b=0$. Then in order that (0, 1, 0) may also be on $A^3x^4y^2$ we must have

$$b_0(b_0g+hn-2lm)=0.$$

If $b_0=0$, the intersection is the contact of a tangent from the double point; there are six such. If $b_0\neq 0$, let us take another intersection which is not one of these contacts as (0, 0, 1). Then

$$c=0,\quad b_0g+hn-2lm=0,\quad c_0h+gm-2ln=0.$$

These last two equations say that the tangents at the double point, given by $hx_1^2+2lx_1x_2+gx_2^2=0$, are apolar to the three points in which x_0 cuts the polar cubic of the double point, these three points being given by

$$b_0\xi_2^3-3m\xi_1\xi_2^2+3n\xi_1^2\xi_2-c_0\xi_1^3=0.$$

But this is true when x_0 is the flex line of the polar cubic. This result can be verified by starting out merely with $a=a_1=a_2=0$ and throwing the polar cubic of (1, 0, 0),

$$3hx_0x_1^2+6lx_0x_1x_2+3gx_0x_2^2+b_0x_1^3+3mx_1^2x_2+3nx_1x_2^2+c_0x_2^3,$$

into the canonical form by

$$h=g=m=n=0.$$

Then x_0 is the flex line. It meets $A^3x^4y^2$ in points given by

$$-l^2(bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4)=0,$$

evidently the same points as those in which x_0 meets the quartic. Therefore the four remaining intersections of $A^3x^4y^2$ and $(ax)^4$ are on the flex line of the polar cubic of the double point.

This result can be verified algebraically. For

$$\begin{aligned}A^3x^4y^2=&(gh-l^2)(6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3+12mx_0x_1^2x_2\\&+12nx_0x_1x_2^2+4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4)\\&-[3hx_0x_1^2+6lx_0x_1x_2+3gx_0x_2^2+b_0x_1^3+3mx_1^2x_2+3nx_1x_2^2+c_0x_2^3]\\&[4(gh-l^2)x_0+(b_0g+hn-2lm)x_1+(c_0h+gm-2ln)x_2]\\=&(gh-l^2)(ax)^4-(ax)^3(ay)\cdot\text{flex line.}\end{aligned}$$

The polar cubic picks up the six intersections at the double point and those at the six contacts of tangents from the double point; the flex line gives the remainder.

The flex line can be obtained by a sort of limit process from the polar lines of $(ax)^4$ and $(hx)^6$. Instead of at once taking (1, 0, 0) as a double point, let us make x_0 the polar line of (1, 0, 0) so that $a_1=a_2=0$. Then if y is (1, 0, 0),

$$\begin{aligned}&3(ay)^4\cdot(hx)(hy)^5+(hy)^6\cdot(ax)(ay)^3\\&\qquad=a[4(gh-l^2)x_0+(b_0g+hn-2lm)x_1+(c_0h+gm-2ln)x_2].\end{aligned}$$

If we divide this by $(ay)^4=a$ and then let $a=0$ so as to get a double point, we have the flex line.

$A^3x^4y^2$ is not the only curve which picks up the six points of contact from y when y is a double point. So does $(hx)^5(hy)$, for when the double point is taken as (1, 0, 0),

$$\begin{aligned}(hx)^5(hy) = &\overset{2}{\Sigma}x_0^3x_1^2\cdot 2(-gh^2+hl^2)+x_0^3x_1x_2\cdot 4(-ghl+l^3)\\
&+\overset{2}{\Sigma}x_0^2x_1^3\cdot(-b_0gh-2b_0l^2-3h^2n+6hlm)\\
&+\overset{2}{\Sigma}x_0^2x_1^2x_2(-3c_0h^2-6b_0gl+3ghm+6l^2m)\\
&+\overset{2}{\Sigma}x_0x_1^4\cdot\tfrac{1}{3}(bgh-4bl^2-b_0^2g-4b_0hn+6b_2hl+2b_0lm-3fh^2+3hm^2)\\
&+\overset{2}{\Sigma}x_0x_1^3x_2\cdot\tfrac{2}{3}(-3bgl-2b_0c_0h+5b_2gh-3c_1h^2-2b_0gm-2b_0ln\\
&\quad -2b_2l^2+3fhl+6lm^2)\\
&+x_0x_1^2x_2^2\cdot\tfrac{1}{3}(-3bg^2-3ch^2-6b_0c_0l-6b_0gn-6b_2gl-6c_0hm\\
&\quad -6c_1hl+18fgh+18lmn)\\
&+\overset{2}{\Sigma}x_1^5\cdot\tfrac{1}{3}(bhn-2blm-b_0^2n+2b_0b_2l-b_0fh+b_0m^2)\\
&+\overset{2}{\Sigma}x_1^4x_2\cdot\tfrac{1}{3}(bc_0h-2bgm-2bln-b_0^2c_0+2b_0b_2g-2b_0c_1h+4b_0fl\\
&\quad +4b_2hn-2b_0mn-2b_2lm-3fhm+3m^3)\\
&+\overset{2}{\Sigma}x_1^3x_2^2\cdot\tfrac{1}{3}(-cb_0h-3bgn+4b_2c_0h-4b_0c_0m+2b_0c_1l+5b_0fg\\
&\quad -2b_2gm-6c_1hm-2b_0n^2-2b_2ln+3fhn+6m^2n),\end{aligned}$$

and the coefficient of x_1^5 vanishes for $b=b_0=0$. That it passes through the intersections of $(\alpha x)^4$ and $(\alpha x)^3(\alpha y)$ is shown by

$$\begin{aligned}3(hx)^5(hy) = &[6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3+12mx_0x_1^2x_2+12nx_0x_1x_2^2\\
&+4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4]\\
&[4(gh-l^2)x_0+(b_0g+hn-2lm)x_1+(c_0h+gm-2ln)x_2]\\
&+[3hx_0x_1^2+6lx_0x_1x_2+3gx_0x_2^2+b_0x_1^3+3mx_1^2x_2+3nx_1x_2^2+c_0x_2^3]\\
&[x_0^2\cdot -10(gh-l^2)+\overset{2}{\Sigma}x_0x_1(-5b_0g-5hn+10lm)\\
&\quad +\overset{2}{\Sigma}x_1^2(-bg-b_0n+2b_2l-fh+m^2)\\
&\quad +x_1x_2(-b_0c_0-2b_2g-2c_1h+4fl+mn)]\\
= &(\alpha x)^4\cdot\text{flex line}-(\alpha x)^3(\alpha y)\left[\frac{1}{a}A^3x^2y^4+x_0\cdot\text{flex line}\right],\end{aligned}$$

where by $\frac{1}{a}A^3x^2y^4$ is meant the result of forming $A^3x^2y^4$ for $y\equiv(1, 0, 0)$ when only $a_1=a_2=0$, dividing by a, and then letting $a=0$. Any further pursuit of this relation, however, when the quartic has no double point, seems merely to lead to Cayley's indentity

$$\begin{aligned}3(\alpha x)^4\cdot(hx)(hy)^5-(\alpha x)^3(\alpha y)\cdot A^3x^2y^4+(\alpha x)(\alpha y)^3\cdot A^3x^4y^2\\
-3(\alpha y)^4\cdot(hx)^5(hy)\equiv 0.\end{aligned}$$

Note 1.

The developments of H and S are given by Salmon only for a special quartic. Those for the general quartic may be calculated by means of a differential operator from the leading coefficients, which in case of S is obtained directly from the invariant S of the cubic as given in the *Higher Plane Curves*. The general expression for the Hessian is

$$\begin{aligned}
&\sum^{3} x_0^6(agh-al^2-a_1^2g-a_2^2h+2a_1a_2l)\\
&+\sum^{6} x_0^5x_1\cdot 2(ab_0g+ahn-2alm-a_2^2b_0-a_1^2n+2a_1a_2m-a_1gh+a_1l^2)\\
&+\sum^{6} x_0^4x_1^2(abg-ba_2^2+4ab_0n-2ab_2l+afh-4am^2+2a_1a_2b_2-a_1^2f+2a_1b_0g-6a_2b_0l\\
&\quad -4a_1hn+6a_2hm+2a_1lm-3gh^2+3hl^2)\\
&+\sum^{3} x_0^4x_1x_2\cdot 2(2ab_0c_0+ab_2g+ac_1h-2afl-2amn-a_1^2c_1-a_2^2b_2+2a_1a_2f-2a_1c_0h\\
&\quad -2a_2b_0g+4a_1gm+4a_2hn-2a_1ln-2a_2lm-3ghl+3l^3)\\
&+\sum^{3} x_0^3x_1^3\cdot 2(abn+ab_0f+ba_1g-2ab_2m-2ba_2l+2a_2b_2h+2a_1b_0n-a_1fh-b_0gh\\
&\quad -2a_1m^2-2b_0l^2-3h^2n+6hlm)\\
&+\sum^{6} x_0^3x_1^2x_2\cdot 2(abc_0+2ab_0c_1-ba_2g-3afm+2a_1b_0c_0+4a_1b_2g-2a_1c_1h+2a_2b_0n\\
&\quad -4a_2b_2l+5a_2fh-3c_0h^2-2a_1fl-6b_0gl-2a_1mn-2a_2m^2+3ghm+6l^2m)\\
&+x_0^2x_1^2x_2^2(abc+2ab_2c_1+2ba_2c_0+2ca_1b_0-3af^2-3bg^2-3ch^2+10a_1b_2c_0+10a_2b_0c_1\\
&\quad -6a_1c_1m-6a_2b_2n-6b_0c_0l-6a_1fn-6a_2fm-6b_0gn-6b_2gl-6c_0hm-6c_1hl\\
&\quad +18fgh+18lmn).
\end{aligned}$$

The full development for S is

$$\begin{aligned}
&\sum^{3} x_0^4(ab_0c_0l-ab_0gn-ac_0hm+agm^2+ahn^2-almn-a_1a_2b_0c_0+a_1^2c_0m+a_2^2b_0n\\
&\quad +a_1b_0g^2+a_2c_0h^2-a_1c_0hl-a_2b_0gl-a_1^2n^2+a_1a_2mn-a_2^2m^2+a_1ghn+a_2ghm\\
&\quad -3a_1glm-3a_2hln+2a_1l^2n+2a_2l^2m-g^2h^2+2ghl^2-l^4)\\
&+\sum^{6} x_0^3x_1(abc_0l-abgn-ba_1a_2c_0-ab_2c_0h+ab_0c_1l+ba_2^2n-ab_0fg+ba_1g^2+2ab_2gm\\
&\quad -ac_1hm-ba_2gl-ab_2ln+2afhn-aflm+a_1^2b_2c_0-a_1a_2b_0c_1+a_2^2b_0f+a_2b_0c_0h\\
&\quad +a_1^2c_1m+a_1a_2b_2n-2a_2^2b_2m-a_1b_0c_0l+a_2b_2gh+a_2c_1h^2-2a_1^2fn+a_1a_2fm\\
&\quad +2a_1b_0gn-3a_1b_2gl-a_1c_1hl+2a_2b_2l^2-2a_2b_0ln+a_1fgh-b_0g^2h-3a_2fhl\\
&\quad +2a_1fl^2-2a_1gm^2-a_2hmn+b_0gl^2+a_1lmn+2a_2lm^2-gh^2n+2ghlm\\
&\quad +hl^2n-2l^3m)\\
&+\sum^{3} x_0^2x_1^2(abc_0m+abc_1l-abfg-abn^2-ab_0b_2c_0-ba_1a_2c_1+ab_2^2g-ab_2c_1h+ba_2^2f\\
&\quad -ba_2c_0h+ab_0fn-ab_2fl+ba_1gn-ba_2gm+ab_2mn+ba_2ln+af^2h+bg^2h\\
&\quad -afm^2-bgl^2+a_1^2b_2c_1-a_2^2b_2^2+a_2b_0^2c_0+a_1a_2b_2f+a_1b_2c_0h+a_2b_0b_2g+a_2b_0c_1h\\
&\quad -a_1b_0c_0m-a_1b_0c_1l-a_1^2f^2-b_0^2g^2-a_1b_2gm-a_2b_0fl+2a_2b_2hn+2a_1b_0n^2-3a_1b_2ln\\
&\quad -3a_2b_0mn-a_1fhn-2a_2fhm-b_0ghn-2b_2ghl+4a_1flm+4b_0glm-a_1m^2n\\
&\quad +2a_2m^3-b_0l^2n+2b_2l^3+fgh^2-fhl^2-ghm^2-h^2n^2+4hlmn-3l^2m^2)
\end{aligned}$$

$$+\sum^{3} x_0^2 x_1 x_2 (abcl - bca_1a_2 - abc_1g - acb_2h + ba_2^2c_1 + ca_1^2b_2 - 2ab_0c_0f + ba_1c_0g$$
$$+ ca_2b_0h + 2ab_0c_1n + 2ab_2c_0m - ba_2c_0l - ca_1b_0l + ab_2fg + ac_1fh - 2ab_2n^2$$
$$- 2ac_1m^2 - af^2l + 2afmn - a_1^2c_1f - a_2^2b_2f - a_1b_0c_1g - a_2b_2c_0h + 2a_1b_0c_0n - a_1b_2c_0l$$
$$+ 2a_2b_0c_0m - a_2b_0c_1l + a_1a_2f^2 + 2a_1c_0fh + 2a_2b_0fg - b_0c_0gh - a_1b_2gn - 2a_1c_1hn$$
$$- 2a_2b_2gm - a_2c_1hm - 4a_1c_0m^2 + 5a_1c_1lm - 4a_2b_0n^2 + 5a_2b_2ln - b_0c_0l^2 + 2b_2g^2h$$
$$+ 2c_1gh^2 + a_1fgm + a_2fhn - 2b_0g^2m - 2c_0h^2n - 3a_1fln - 3a_2flm + 4b_0gln$$
$$- 2b_2gl^2 + 4c_0hlm - 2c_1hl^2 + 2a_1mn^2 + 2a_2m^2n - 4fghl - 3ghmn + 4fl^3$$
$$+ 4glm^2 + 4hln^2 - 7l^2mn).$$

The leading coefficient of T may be obtained from Salmon's expression for the invariant T of the cubic, but the coefficients are too tedious to compute in general. For the coefficients of s and t see AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIX (1917), p. 232.

Note 2.

Certain invariants of the quartic have a well-defined geometrical meaning. The two simplest invariants, the A^3 and A^6, have apolarity meanings and may be included in the list by courtesy. Then comes the A^{15} of Professor Coble,* which is the condition that the quartic be reducible to the sum of the fourth powers of the six lines of a complete quadrilateral, also that the covariant S become two conics; the discriminant, A^{27}; the A^{48} of Dr. Thomsen, expressing that there be a polar conic which is the square of a line; an A^{54} of Professor Morley's expressing the condition that the quartic pass through the vertices of a pentagon; the undulation condition, A^{60}. In this list certain gaps are filled in by the A^{45}, the condition that a polar cubic be made up of three lines, and the A^{51}, the condition that the polar conic of two points be the square of their join.

There should also be an A^{24} under which the Steinerian has the stationary lines of the quartic as double lines. For the Steinerian in lines, an $A^{36}\xi^{18}$, and the Cayleyan, an $A^{12}\xi^{18}$, touch on the stationary lines.† Then the terms of the Steinerian containing only the first power of $(s\xi)^4$ or $(t\xi)^6$ should be the same as those of the Cayleyan, multiplied by an A^{24} to bring them up to the proper degree. If this A^{24} vanishes, then Σ has the stationary lines as double lines.

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* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI (1909), p. 357.

† Proc. Nat. Ac. Sci., Vol. III (1917), p. 449.

On Surfaces Containing Two Pencils of Cubic Curves.

By C. H. Sisam.

1. In a recent article in the American Journal of Mathematics, Vol. XLI, p. 49, the author has discussed the surfaces generated by an algebraic system of cubic curves that do not constitute a pencil. In this article he studies the surfaces which contain two or more pencils of cubic curves.

2. Let m be the order of a given surface F which is generated by two pencils Σ_1 and Σ_2 of cubics which intersect in k variable points. We shall show that

$$mk \leq 18.$$

Any surface of order kx such that $kx > m$ contains all the curves of Σ_2, and hence has F as a component, if it contains $3x+1$ generic curves of Σ_1. Hence it contains F if it contains $3kx+1$ generic points on each of $3x+1$ generic curves of Σ_1, that is, if it satisfies at most, $(3x+1)(3kx+1)$ independent linear conditions. Then this number can not be less than the actual number of conditions that a surface of order kx must satisfy in order to contain F as a component, that is

$$(3x+1)(3kx+1) \geq \frac{(kx+3)!}{3!\,kx!} - \frac{(kx-m+3)!}{3!\,(kx-m)!},$$

or

$$3k(18-mk)x^2 + (3km^2-12km+18k+18)x - (m^3-6m^2+11m-6) \geq 0.$$

Since this inequality holds for all integer values of x such that $xk > m$, we must have $18-mk \geq 0$.

We have supposed, in the above proof, that F lies in three dimensions. But the theorem still holds if F belongs to $r > 3$ dimensions, since such a surface can be projected, without changing m or k, into one belonging to three dimensions.

Case I. The Cubics of Both Given Systems are Rational.

3. In this case the surface F is rational since it contains two pencils of rational curves. The entities Σ_1 and Σ_2, whose elements are the curves of the

two systems, are also rational since each defines a rational involution of order k on a generic curve of the other pencil.

$$k=1.$$

4. Let the pencils Σ_1 and Σ_2 be put in (1, 1) correspondence with pencils of lines in a plane Π having vertices at P_1 and P_2 respectively. Then the surface F and the plane Π are in (1, 1) correspondence in such a way that corresponding points are at the intersections of corresponding curves. Let μ be the order of the curves in Π that correspond to the hyperplane sections of F. Since these curves have points of multiplicity $\mu-3$ at P and P_1' we have $2(u-3)\leq\mu$ or $\mu\leq 6$.

5. The system of sextic curves in Π which have triple points at P and P' define parametrically a surface F which is of the given type. Any other such surface is a projection of this one, since any linear system of curves of order μ having $(\mu-3)$-fold points at P and P' is contained in the linear system of sextics. We conclude that if the curves of Σ_1 and Σ_2 are rational and $k=1$, *the surface is of order* 18, *belongs to an* S_{15}* *and is defined parametrically by the sextics in a plane which have two basis triple points or it is a projection (not necessarily from external points) of such a surface.*

6. If F belongs to S_{15}, the genus of a generic hyperplane section is equal to 4, that is, to the genus of a generic sextic with two triple points. It contains no cubic curves other than those of the two given systems.

$$k=2.$$

7. Let F be birationally transformed into the plane Π in such a way that the cubics of Σ_2 correspond to the lines through P'. Then the cubics of Σ_1 correspond to rational curves of order ν which have a $(\nu-2)$-fold point at P'. By a suitable birational transformation the order of the curves of this pencil can be reduced to 2. Let, then, $\nu=2$ and let P_1, P_2, P_3, P_4 be the basis points of this pencil of conics. To the hyperplane sections of F correspond curves of order μ which have a $(\mu-3)$-fold point at P' and multiplicities ρ_i at P_i, $(i=1, 2, 3, 4)$ such that $2\mu-(\rho_1+\rho_2+\rho_3+\rho_4)=3$. By a suitable birational transformation, such a system of curves can be reduced to one of the following three types:

(1) $\mu=4$ $\rho_1=2$ $\rho_2=\rho_3=\rho_4=1$.
(2) $\mu=3$ $\rho_1=0$ $\rho_2=\rho_3=\rho_4=1$.
(3) $\mu=3$ $\rho_1=2$ $\rho_2=1$, $\rho_3=\rho_4=0$.

* The symbol S_r denotes a space of r dimensions.

8. The surfaces defined parametrically by systems of types (2) and (3) are projections of special cases of surfaces of type (1) such that the latter surface has one or more double points. Hence, if the cubics of Σ_1 and Σ_2 are rational and $k=2$, *the surface is of order* 8, *belongs to an* S_7 *and is defined parametrically by the quartics in a plane which have a double and four simple basis points, or it is a projection of such a surface.*

9. If F belongs to S_7, its generic hyperplane sections are of genus 2. It contains four pairs of pencils of cubics such that cubics of opposite systems intersect in two points. One pencil of such a pair is defined by the pencil of conics through the double and three simple basis points. The other pencil is defined by the lines through the remaining basis points. It contains no other cubic curves.

$$k=3.$$

10. As in the preceding case, this surface can be transformed into a plane Π in such a way that the cubics of Σ_2 correspond to the lines through P' and the cubics of Σ_1 to the cubics which have a basis double point P_1 and five basis simple points $P_2, P_3, \ldots, P_6$. To the hyperplane sections of F correspond curves of order μ which have a $(\mu-3)$-fold point at P' and multiplicities ρ_i at P_i, $(i=1, 2, 3, \ldots, 6)$ such that $3\mu-(2\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6)=3$. If the order m of F exceeds three, these curves can be transformed into quartics which have a double point at P_1 and pass simply through $P_2, P_3, \ldots, P_6$ and P'. If $m\leq 3$ the surface is a projection of one with one or more singular points defined by a system of the above type. Hence, if the cubics of Σ_1 and Σ_2 are rational and if $k=3$, *the surface is of order* 6, *it belongs to an* S_5 *and is defined parametrically by the quartics in a plane which have a double and six simple basis points, or it is a projection of such a surface.*

11. There are thirty-two pencils of cubic curves on such a surface F_1 defined by the pencils of curves of order $i\leq 3$ in Π which have an $(i-1)$-fold point at P_1 and pass through $2i-1$ simple basis points. The curves of each pencil intersect those of one other pencil in three points. The S_3 defined by the curves of such a pair of pencils constitute the two systems of S_3 on an hyperquadric in S_5 which has a double line and contains F.

There are twelve other cubic curves on F. Six are defined by the conics through five simple basis points and the rest by the cubics which have a node at one simple basis point and pass through all the other basis points.

$$k\geq 4.$$

12. If $k=4$ and if $m=4$, the surface F can be transformed into Π in such a way that the curves Σ_2 correspond to the lines through P', the curves

Σ_1 to quartics with a basis double point P_1 and seven basis simple points $P_2, P_3, \ldots, P_8$, and that the plane sections of F correspond to quartics which have a double point at P_1 and pass through $P_2, P_3, \ldots, P_8$ and P'. Such a linear system defines a quartic F with a double line.

We have seen (Art. 2) that if $k>4$, then $m\leq 3$. Hence, *if* $k\geq 4$, *the surface* F *is a quartic with a double line, or a cubic, a quadric or a plane.*

CASE II. THE CUBICS OF Σ_1 ARE RATIONAL; THOSE OF Σ_2, ELLIPTIC.

$k=1$.

13. The pencil Σ_2 is rational since it is in (1, 1) correspondence with a generic curve of Σ_1. Similarly, Σ_1 is elliptic. Let the pencil Σ_1 be put in (1, 1) correspondence with the pencil of rectilinear generators of a ruled quintic surface ϕ belonging to S_4 and let the pencil Σ_2 be put in (1, 1) correspondence with a pencil Σ' of quartics on ϕ having three basis points P_1, P_2, P_3. Then F and ϕ are in (1, 1) correspondence in such a way that corresponding points are at the intersections of corresponding curves.

14. Denote by ρ_1, ρ_2, ρ_3 the multiplicities at P_1, P_2, P_3 of the curves on ϕ which correspond to the hyperplane sections of F. The curves of this linear system intersect the rectilinear generators in three points and the quartics of Σ' in $3+\rho_1+\rho_2+\rho_3$ points. Their order $6+\rho_1+\rho_2+\rho_3$ is found by counting their intersections with an hyperplane containing a quartic of Σ' and a generator. These curves are cut from ϕ by a linear system of quintic hypersurfaces H^5 which have contact of second order with ϕ at P_1, P_2 and P_3, and contain as basis curves (a), the generators through P_1, P_2, P_3 counted respectively $3-\rho_1, 3-\rho_2, 3-\rho_3$ times, (b) a quintic curve C^5 which intersects the generators twice and (c), five generators of ϕ. For, since C^5 is elliptic, in a linear system of seventy-four dimensions * of H^5 that do not contain ϕ, there exists a linear system of thirty-one dimensions of H^5 which contain C^5 and have contact of second order at P_1, P_2 and P_3. The curves corresponding to the hyperplane sections of F intersect these H^5 in $3(\rho_1+\rho_2+\rho_3)$ points at P_1, P_2, P_3; in $2(\rho_1+\rho_2+\rho_3)-3$ points on C^5 and in thirty-three other points. These curves are of genus 3 (at least) since F is neither ruled nor rational,† hence each curve of the system lies on such an H^5. The residual intersection does not intersect a generic generator. Hence it degenerates into the generators through P_1, P_2, P_3 counted $3-\rho_1, 3-\rho_2$ and $3-\rho_3$ times and five generators.

* Cf. the Author, AMERICAN JOURNAL OF MATHEMATICS, Vol. XLI, p, 51.

† Cf. Castelnuovo-Enriques, *Mathematische Annalen*, Vol. XLVIII, p. 308.

The linear system of curves on ϕ which corresponds to the hyperplane sections of F is thus contained in (or coincides with) a linear system L of curves of order 15 and genus 7, having triple points at P_1, P_2, P_3 which is cut from ϕ by a linear system of H^5 which contain C^5 and five fixed generators, and have contact of second order with ϕ at P_1, P_2, P_3. It follows that: *the surface F is of order* 18, *it belongs to a space of eleven dimensions and its generic hyperplane sections are of genus* 7, *or it is a projection of such a surface.*

15. Let the correspondence between Σ_1 and the generators of ϕ be set up in such a way that the g_3^2 defined on a given generic cubic C of Σ_2 corresponds to the g_3^2 defined on the quartic corresponding to C by the hypercubics having second order contact with ϕ at P_1, P_2, P_3. Then the linear system L is cut from ϕ by this system of hypercubics, that is: *the surface F can be birationally represented on a ruled quintic belonging to S_4 in such a way that the curves corresponding to the hyperplane sections of F are cut from the quintic by cubic hypersurfaces which have second order contact with the quintic at the basis points of a pencil of quartic curves on it.*

16. Let F belong to S_{11}. Its parametric equations can be reduced to the form

$$\begin{array}{llll} x_0=1 & x_1=v & x_2=v^2 & x_3=v^3 \\ x_4=\wp(u) & x_5=v\wp(u) & x_6=v^2\wp(u) & x_7=v^3\wp(u) \\ x_8=\wp'(u) & x_9=v\wp'(u) & x_{10}=v^2\wp'(u) & x_{11}=v^3\wp'(u), \end{array}$$

wherein $\wp(u)$ is the Weierstrassian $\wp$-function.

The residual section of F by an hyperplane which contains the planes of three cubics of Σ_2 degenerates into three curves of Σ_1 since it intersects a generic curve of Σ_2 in three points and has no points in common with a generic curve of Σ_1. Three such cubics of Σ_1 lie in the S_7 defined by the S_8 of two of them, since every hypersurface that contains two of them contains the third.

17. The order of any curve on F that intersects the curves of Σ_1 in x points, and those of Σ_2 in y points is

$$n=3(x+y),$$

as may be seen by counting its intersections with a generic hyperplane which intersects F in three cubics of each system. It follows that the order of every curve on F is a multiple of three. Moreover, since there are no rational curves on F except those of Σ_1, and since every curve C on F for which $y=1$ is in (1, 1) correspondence with Σ_2 and hence is rational, we conclude that *there are no curves of order less than nine on the surface F belonging to S_{11}*

except those of Σ_1 *and* Σ_2. Curves of order 9 on F actually exist. Such a curve is defined by a generic cubic on ϕ, or by a generic quartic through P_1, P_2 or P_3.

$$k=2.$$

18. The involution γ_2^1 of order 2 defined by Σ_1 on a generic cubic C of Σ_2 is rational. For, if γ_2^1 is elliptic, it has no double points, so that no curve of Σ_1 has its two intersections with C coincident. But the involution defined by Σ_2 on a generic curve of Σ_1 has two double points. Hence each of two fixed curves of Σ_2 would degenerate into a curve counted twice. These degenerate curves are of genus 1 since they are in (1, 1) correspondence with Σ_1. But this is impossible, since the curves of Σ_2 are cubics. Then γ_2^1 is rational, Σ_1 is rational, and F is rational since it contains a rational pencil of rational curves.

19. The surface F can be transformed into a plane Π in such a way that to Σ_1 corresponds a pencil of lines through P' and to Σ_2, a pencil of cubics through P' and eight other points $P_1, P_2, \ldots, P_8$. The linear system of curves of order μ corresponding to the hyperplane sections of F can be reduced to a system which coincides with, or is contained in the complete linear system of quartics which pass through P' and have multiplicities ρ_i at P_i such that either

$$(1) \quad \rho_1=\rho_2=\ldots=\rho_8=1,$$

or

$$(2) \quad \rho_1=2,\ \rho_2=\ldots=\rho_7=1,\ \rho_8=0.$$

Hence, if a generic cubic of Σ_1 is rational, and of Σ_2 elliptic, and if $k=2$, then either

(1) *The surface is of order* 7, *it belongs to an* S_5 *and is defined parametrically by the quartics in a plane through the nine basis points of a pencil of cubics, or it is a projection of such a surface,* or (2) *The surface is of order* 5, *it belongs to an* S_4 *and is defined parametrically by the quartics in a plane with a double and seven simple basis points, or it is a projection of such a surface.*

On the surface (1), there are in general eight other pencils of rational cubics which intersect the cubics of Σ_2 in two points and 126 cubics which intersect the cubics of Σ_2 once. On the surface (2) there are in general, sixty-three other pencils of rational cubics which intersect the cubics of Σ_2 twice, and eighty-six cubics which intersect those of Σ_2 once.

$$k=3.$$

20. As in the preceding case, the involution γ_3^1 defined by the curves of Σ_1 on a generic curve of Σ_2 is rational. For, if not, it has no double points,

while the involution defined by Σ_2 on a generic curve of Σ_1 has six double points. Then six curves of Σ_2 have as double component a curve of genus 1, which is impossible.

It follows that F is rational. It can be represented on a plane Π in such a way that Σ_1 corresponds to a pencil of lines through P', Σ_2 to a pencil of cubics through $P_1, P_2, \ldots, P_9$ and the hyperplane sections to a system of curves which coincide with, or are contained in the complete system of quartics which pass through P' and have multiplicities ρ_i at P_i such that either

$$(1) \quad \rho_1=\rho_2=\ldots=\rho_9=1,$$

or

$$(2) \quad \rho_1=2,\ \rho_2=\rho_3=\ldots=\rho_8=1,\ \rho_9=0.$$

Hence, if a generic cubic of Σ_1 is rational and of Σ_2, elliptic, and if $k=3$, then either

(1) *The surface is of order* 6, *belongs to an* S_4 *and is defined parametrically by the quartics through ten points of which nine are the basis points of a pencil of cubics, or it is a projection of such a surface,* or (2) *The surface is a quartic with a nodal line. It belongs to* S_3.

21. Each of these surfaces is the projection of one of the surfaces of Art. 19 from a point P on the surface. Each has a double line l which is the projection of the elliptic cubic through P. The cubics of Σ_1 are the projections of a pencil of quartics through P. On the surface (1), the cubics of Σ_1 have l as a locus of double points. The planes of the cubics of Σ_1 and Σ_2 on this surface in S_4 generate a singular hyperquadric which has F as its complete intersection with an hypercubic.

CASE II. THE CUBICS OF BOTH SYSTEMS ARE ELLIPTIC.

$$k=1.$$

22. We can write at once the parametric equations of a surface which clearly belongs to the required type. Let $\wp(u)$ and $\bar{\wp}(v)$ be two Weierstrassian $\wp$-functions (having in general different moduli). Then the surface

$$\left.\begin{array}{lll} x_0=1 & x_1=\wp(u) & x_2=\wp'(u) \\ x_3=\bar{\wp}(v) & x_4=\wp(u)\bar{\wp}(v) & x_5=\wp'(u)\bar{\wp}(v) \\ x_6=\bar{\wp}'(v) & x_7=\wp(u)\bar{\wp}'(v) & x_8=\wp'(u)\bar{\wp}'(v) \end{array}\right\} \qquad (1)$$

is generated by two pencils of elliptic cubics $u=$const. and $v=$const. such that cubics of opposite systems intersect in one point.

23. Any two cubics of one system on (1) lie in an S_5 which contains a third cubic of the same system. The residual section of the surface by an

hyperplane that contains such an S_5 is composed of three cubics of the opposite system. Hence, the order of this surface is 18 and, since cubics of the same system do not intersect, the genus of a generic hyperplane section is equal to 10.* It follows further that a curve on the surface which intersects the curves $u=$const. in x-points and the curves $v=$const. in y-points is of order $n=3(x+y)$ as may be seen by counting the intersections of the curve with an hyperplane which intersects (1) in six cubics. If either $x=1$, $y>0$ or $y=1$, $x>0$ there is a $(1, y)$ or $(1, x)$ correspondence between the two systems of cubics, and hence a particular relation between the moduli. Hence, *in general there are no curves of order less than* 12 *on the surface* (1) *except the cubics* $u=$*const. and* $v=$*const.*

24. Let F be a given surface generated by two pencils Σ_1 and Σ_2 of elliptic cubic curves that intersect in one point. Let the moduli of the pencils on (1) be respectively equal to those of Σ_1 and Σ_2 and let the pencils $u=$const. and Σ_1 and $v=$const. and Σ_2 be put in (1, 1) correspondence. Then the surfaces (1) and F are in (1, 1) correspondence in such a way that corresponding points are at the intersection of corresponding curves. To an hyperplane section of F corresponds on (1) a curve C^{18} which is of order 18 since it intersects the curves $u=$const. and $v=$const. in three points.

25. In the S_8 defined by the surface (1), there exists a linear system of $9n^2-1$ dimensions of hypersurfaces H^n of order n which do not contain the surface (1) since such an H^n can be made to contain $3n-1$ generic curves $u=$const. and $3n-1$ generic points of an additional curve $u=$const. without containing (1). If n is sufficiently large, there are, in such a linear system, at least $3n-12$ linearly independent H^n which contain a given C^{18} and $3n-4$ given generic cubics $u=$const. The residual intersection of all these H^n with any given generic cubic $v=$const. is a fixed point P. Hence they all contain a fixed curve C on (1) which intersects the cubics $v=$const. in one point and the cubics $u=$const. in a certain number $y\geq 0$ of points. Their residual intersections with (1) have no points in common with a generic curve $v=$const. and break up into $3n-3-y$ curves of that pencil. The curve C is of order $3(1+y)$. Its genus is unity since it intersects the curves $v=$const. once. In general, $y=0$ (Art. 23).

26. If $y=0$, the curve C is a cubic of $u=$const. and the curves C^{18} corresponding to the hyperplane sections of F are defined on (1) by a linear system of H^n through $3n-3$ fixed cubics of each system. The complete linear system to which they belong is of dimension $9n^2-1-[3n(3n-3)+3(3n-3)]=8$.

* Cf. Castelnuovo-Enriques, *Annale de Matematica*, Ser. 3, Vol. VI, p. 171.

Let

$$u_1, u_2, \ldots, u_{3n-3} \text{ and } v_1, v_2, \ldots, v_{3n-3}$$

be the parameters of the $6n-6$ basis cubics of the given system of H^n. The parameters u, u', u'' of the three intersections with any cubic $v=$const. of a C^{18} corresponding to an hyperplane section of F satisfy the congruence

$$u+u'+u'' \equiv -(u_1+u_2+\ldots+u_{3n-3}) \mod (\omega_1\omega_2).$$

Similarly, the parameters v, v', v'' of its intersections with a cubic $u=$const. satisfy

$$v+v'+v'' \equiv -(v_1+v_2+\ldots+v_{3n-3}) \mod (\bar{\omega}_1\bar{\omega}_2).$$

Let the correspondence between F and (1) be set up in such a way that the g_3^2 defined by the lines in the plane of one given cubic of each system on F is transformed into the g_3^2 defined by the lines in the plane of the corresponding cubic on (1). Then each member of the above congruences is congruent to zero, and the complete linear system of eight dimensions, to which the curves C^{18} belong is cut from (1) by a linear system of H^n which degenerate into a fixed H^{n-1} containing the $6n-6$ cubics, together with the system of hyperplanes of S_8. We conclude that *in general,* $y=0$ *and the surface* F *coincides with a surface* (1) *or with a projection of such a surface.*

27. If $y>0$, the curves corresponding to the hyperplane sections of F are cut from (1) by a linear system of H^n which contain $3n-4$ fixed cubics $u=$const., $3n-3-y$ fixed cubics $v=$const., and a fixed curve C of order $3(1+y)$ and genus 1. The complete linear system L of curves cut from (1) by the H^n which contain the above curves is of dimension

$$9n^2-1-[3(1+y)n+(3n-4)(3n-y)+(3n-3-y)3]=8-y.$$

Since the dimension of this system can not be less than 3, we have $y \leq 5$.

The curve C intersects the curves of the system L in $3+5y$ points. In fact, by varying the basis curves $u=$const. we can obtain a pencil of curves C (no two of which intersect) each of which, with fixed curves $u=$const. and $v=$const., forms the system of basis curves of a linear system of H^n that define L. Each curve C intersects an H^n of a system which does not contain it in $3n(1+y)$ points of which $3n(1+y)-5y-3$ lie on the basis cubics and $5y+3$ on a curve of L. It follows that the order of the system L and hence of the surface F, is

$$m=18n-(5y+3)-3(6n-7-y)=18-2y.$$

Let p be the genus of a generic curve of L. The virtual genus of the residual section of (1) by an H^n of a linear system which defines L is

$9(n-1)^2-y+1$, and of the complete section of (1) by H^n is $9n^2+1$. Then

$$p+9(n-1)^2-y+1+18n-(18-2y)-1=9n^2+1$$

or
$$p=10-y.$$

Hence, *if* $5\geq y\geq 0$, *the surface* F *is of order* $18-2y$, *its hyperplane sections are of genus* $10-y$ *and it belongs to a space of* $8-y$ *dimensions, or it is a projection of such a surface.*

$$k=2.$$

28. If F belongs to an $S_r (r>3)$ let it be projected from an S_{r-4} which does not intersect it onto a surface F' belonging to S_3. Denote the projections of Σ_1 and Σ_2 by Σ_1' and Σ_2'. If the involution γ_2^1 defined on a generic curve C' of Σ_1' by the curves of Σ_2' is elliptic, the lines joining corresponding points envelope a curve of class 3, if it is rational, they constitute a pencil with vertex on C'. Since at most one cubic of Σ_2 can be coplanar with C'—otherwise a generic cubic of Σ' would have four points in common with this plane—the surface F' belongs to one of the following types:

α. The curves of Σ_1' and Σ_2' lie in pairs in the tangent planes of a developable of class 4 and genus 1. This developable can not reduce to a cone since it can not have a double plane.

β. The planes of each system envelope a cone of class 3 and genus 1.

γ. The planes of Σ_1' generate an axial pencil; those of Σ_2' envelope a cone of class 3 and genus 1.

δ. The planes of each system generate an axial pencil.

ε. The curves of Σ_1' and Σ_2' lie in pairs in the tangent planes to a quadric cone.

29. Cases α, β and γ do not exist. In case α, the residual section of F' by a plane of the developable is elliptic since it is intersected by the curves of Σ_1 and Σ_2 in one variable point. It is a cubic since the order of F' does not exceed 9 (Art. 2). Denote this system of cubics by Σ_3'. Two generic cubics of Σ_3' intersect in one variable point since, of the three intersections of each with the plane of the other, only two lie on the curves of Σ_1' and Σ_2'. Hence F' contains a pencil of rational curves* on which the pencil Σ_1' (or Σ_2') defines an elliptic involution. This is impossible.

In case β, let P_1 and P_2 be the vertices of the cones. The planes of three curves of Σ_2' pass through P_1. The system of lines joining corresponding points of the involution on each of these cubics degenerates into a pencil counted three times. Then these curves degenerate and intersect a generic

* Cf. the Author, AMERICAN JOURNAL OF MATHEMATICS, Vol. XLI.

curve of Σ_1 in coincident points. But this is impossible since the involution on a generic curve of S_1 has no double points.

In case γ, let μ be the multiplicity on F' of the axis l of the pencil of planes of Σ_1'. Then the order of F' is $\mu+3$. A generic curve C' of Σ_2' intersects l. The residual section in its plane is of order μ and has a $(\mu-1)$-fold point on l. Hence it is rational. The curves of Σ_2' define an elliptic involution on this rational curve, which is impossible. We now conclude that Σ_1' and Σ_2' are both rational.

30. In case δ, we denote the axes of the pencils by l_1 and l_2 and suppose, first, that they do not intersect. Then l_1 (or l_2) is a simple line on the surface since a generic point of it is a simple point on just one curve of Σ_2' (or Σ_1'). Hence F' is a quartic with two skew simple lines. Since l_1 and l_2 do not intersect, and the cubics of Σ_1 and Σ_2 are elliptic, the quartic F is not the projection of a surface of the same order belonging to more than three dimensions; for, the genus of a generic plane section is less than 2 only if the surface is a ruled quartic of genus 1.

31. If l_1 and l_2 intersect, all the curves of both systems pass through their intersection. Let μ be the multiplicity of l_1 and l_2 on F'. Then one curve of each system degenerates into the axis of the other system counted μ times, together with a residual curve of order $3-\mu$. Thus $\mu \leq 3$ and $m=3+\mu \leq 6$. The intersection of l_1 and l_2 is a $(\mu+1)$-fold point on the surface. The linear system of quadrics which contain l_1 and l_2 defines a birational transformation of F' into a sextic surface F_1 belonging to S_4 and of Σ_1' and Σ_2' into pencils of cubics on F_1, all of which pass through a fixed point P which is a double point on F_1. The planes of these two pencils of cubics constitute the two systems of planes on an hyperquadric H^2 with a double point at P. The surface F_1 lies on (at least) six linearly independent hypercubics of which at most five have H^2 as a component. Hence, it forms the complete intersection of H^2 with an hypercubic. The surface F' is the projection of F_1 from a point on H^2.

32. In case ε, all the cubics of both systems pass through the vertex of the cone which is a double point on F'. The order of F' is 6 since a residual curve in the plane of two coplanar cubics would have to be a component of a curve of each system. Three of the intersections of coplanar cubics lie on a generator of the quadric cone. The other six lie on a nodal sextic which forms the intersection of a cubic and a quadric surface. The linear system of cubic surfaces which contain this nodal sextic define a birational transformation of

F' into a surface of the type of F_1 of Art. 31. The surface F' is the projection of the surface F_1 from a point not lying on the hyperquadric.

33. Every surface F belonging to S_4 is of the type F_1 since it lies on, at most, one hyperquadric, and its projection from a point not on this hyperquadric is a sextic F' of the type of Art. 32. Since no surface F_1 is a projection of a surface of the same order belonging to S_5, we conclude that, if generic cubics of Σ_1 and Σ_2 are elliptic and if $k=2$, then

(1) *If the cubics of both systems do not all pass through a fixed point P, the surface is a quartic, belongs to S_3 and has two skew simple lines.*

(2) *If the cubics of both systems pass through a fixed point P, the surface is the complete intersection of a cubic hypersurface in S_4 with a quadric hypersurface which has a double point at P, or it is a projection of such a surface.*

34. Neither of these surfaces contains, in general, any other pencils of cubics. In fact, their geometric genus is in general unity so that they do not contain a pencil of rational cubics. If the surface (1) contains an additional pencil of elliptic cubics, it contains an additional simple line, if the surface (2) belonging to S_4 contained such a pencil, the planes of the cubics would lie on the hyperquadric.

$$k=3.$$

35. If F belongs to an S_r $(r>3)$ let it be projected from an S_{r-4} that does not intersect it into a surface F' belonging to S_3. The lines of section of the plane of a cubic C' of Σ_1' by the the planes of Σ_2' constitute a pencil since through a generic point of C' there passes just one such line. Hence, either the planes of Σ_2' (and, similarly, of Σ_1') constitute an axial pencil or the curves of the two systems lie in pairs in the tangent planes to a quadric cone.

36. Let the two systems of planes constitute two pencils whose axes l_1 and l_2 do not intersect. Then neither l_1 nor l_2 lies on F' so that F' is a cubic. Since a generic curve of Σ_1 (or Σ_2) is elliptic, this cubic can not have a nodal line and is not the projection of a surface of the same order belonging to S_4.

37. If the axes of the pencils intersect, or if the curves lie in pairs in the tangent planes to a quadric cone, we see as in Arts. 31 and 32, that F' is the projection of a surface F_1 belonging to S_4 which forms the complete intersection of a cubic hypersurface with a quadric hypersurface having a double point which does not lie on the cubic.

38. It follows as in Art. 33, that if generic cubics of Σ_1 and Σ_2 are elliptic and if $k=3$, then

(1) *If the planes of the cubics of both systems do not all pass through the same fixed point, the surface is a cubic which can not have a double line.*

(2) *If the planes of the cubics of both systems pass through the same fixed point, the surface is the complete intersection of a cubic hypersurface in S_4 with a quadric hypersurface which has a double point not lying on the cubic, or it is a projection of such a surface.*

As in Art. 34, it is seen that the surface (2) does not in general contain any other pencils of cubic curves.

39. The results of the foregoing discussion are summarized in the following table. In this table p_1 and p_2 denote respectively the genus of a generic curve of Σ_1 and Σ_2, k is the number of variable intersections of curves of opposite systems; m is the maximum order of a surface F containing Σ_1 and Σ_2; p is the maximum genus of an hyperplane section of F; r is the maximum number of dimensions to which F belongs, and s is the number of distinct or consecutive pencils of cubic curves necessarily existing on the surface F of maximum order. The numbers p_g and p_a are respectively the geometric and arithmetic genus of F except in the last two cases, where they are the maximum values of those numbers.

p_1	p_2	k	m	r	p	s	p_g	p_a
0	0	1	18	15	4	2	0	0
0	0	2	8	7	2	8	0	0
0	0	3	6	5	2	32	0	0
0	0	4	4	3	2	128	0	0
0	1	1	18	11	7	2	0	−1
0	1	2	7	5	3	10	0	0
0	1	3	6	4	3	11	0	0
1	1	1	18	8	10	2	1	−1
1	1	2	6	4	4	2	1	1
1	1	3	6	4	4	2	1	1

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Modular Invariants of a Quadratic Form for a Prime Power Modulus.

By J. E. McAtee.*

Introduction.

We consider a quadratic form q in n variables whose coefficients are integers taken modulo P^λ, where P is an odd prime. Two such forms are said to be equivalent modulo P^λ if they become identically congruent under a linear homogeneous transformation whose coefficients are congruent to integers modulo P^λ and their determinant is congruent to unity. All the forms equivalent to q are said to form a class. A single-valued function ϕ of the coefficients of q is an invariant of q if it has the same value for all forms of the same class. In case the values taken by ϕ are integers which may be reduced modulo P^λ, the invariant is called modular. We obtain a complete invariantive classification of n-ary quadratic forms modulo P^λ and construct modular invariants which completely characterize the classes. Professor Dickson† had treated the problem for the case $\lambda=1$, when the fundamental invariants are the determinant D of q, the rank r modulo P of D, and A_r, to which the invariant $A_{\bar{R}_0}(D)$ of § 4 reduces for $\lambda=1$.

In his investigation of the number of sets of solutions of the congruence $q\equiv c \pmod{P^\lambda}$, C. Jordan‡ obtained from q by a seriatim process of reduction the canonical form

$$P^{e_1}(A_1x_1^2+\ldots.+A_sx_s^2)+P^{e_2}(A_{s+1}x_{s+1}^2+\ldots.+A_tx_t^2)+P^{e_3}()+\ldots,$$

but gave no explicit method to deduce à priori from q the number of terms in each parenthesis or the values of e_1, e_2 ..... This is accomplished by the present invariants.

The second part of the paper is devoted to the polynomial modular invariants of a binary quadratic form modulo P^λ, in particular to the determination of a fundamental system of such invariants modulo 4.

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* Dr. McAtee died at Urbana, Ill., Dec. 2, 1918. The proof-sheets were compared with both the final and an initial draft of the thesis. In § 7, an inadequate proof of Theorem 1 has been replaced by the present shorter proof, while the easy proofs of some other facts have been omitted. The proof of formula (3) of § 1 has been suppressed, as it was entirely similar to that cited. L. E. Dickson.

† Madison Colloquium Lectures, 1913, Ch. 1; *Trans. Amer. Math. Soc.*, Vol. X (1909), p. 123.

‡ *Jour. de Math.*, (2), Vol. XVII (1872), pp. 368–402. Cf. Bachmann, "Die Arith. der Quadratischen Formen" (1898), p. 434.

I. INVARIANTS OF THE n-ARY QUADRATIC FORM MODULO P^λ.

1. *Canonical Forms and Classes.*—Consider the form

$$q_n=\sum_{i,j=1}^{n} a_{ij}x_ix_j \qquad (a_{ij}=a_{ji}), \tag{1}$$

where the a_{ij} are integers taken modulo P^λ, where P is an odd prime. If every a_{ij} is divisible by P^λ, the form (1) is said to constitute the class C_0. In the contrary case, let e_1 be the exponent of the highest power of P that divides every a_{ij}, and write

$$q_n=P^{e_1}\sum_{i,j=1}^{n} A_{ij}^{(1)}x_ix_j, \qquad D=|A_{ij}^{(1)}|. \tag{2}$$

First, let the determinant D be prime to P. Since there exists a primitive root of P^λ, it follows as in the Madison Colloquium Lectures, pp. 6, 7, that q_n can be transformed into

$$P^{e_1}[x_1^2+\ldots.+x_{n-1}^2+Dx_n^2] \tag{3}$$

by a linear transformation with integral coefficients of determinant unity modulo P^λ.

Second, let D be divisible by P, and r_1 its rank modulo P. As on p. 9 of the Lectures cited, we may assume that

$$M_{r_1}=|A_{ij}^{(1)}|\not\equiv 0 \pmod P \qquad (i,j=1,\ldots.,r_1),$$

and show* that q_n can be transformed into

$$P^{e_1}[\sum_{i,j=1}^{n-1} A_{ij}^{(1)}x_ix_j+2\sum_{j=r_1+1}^{n-1} B_{jn}^{(1)}x_jx_n+B_{nn}^{(1)}x_n^2],$$

where

$$B_{jn}^{(1)}\equiv\begin{vmatrix} A_{11}^{(1)}\ldots.A_{1r_1}^{(1)} & A_{1n}^{(1)} \\ \ldots\ldots\ldots\ldots \\ A_{r_11}^{(1)}\ldots.A_{r_1r_1}^{(1)} & A_{r_1n}^{(1)} \\ A_{j1}^{(1)}\ldots.A_{jr_1}^{(1)} & A_{jn}^{(1)} \end{vmatrix} M_{r_1}^{-1} \qquad (j=r_1+1,\ldots.,n)$$

is congruent to an integer modulo P^λ and is divisible by P. Repetitions of this process give

$$P^{e_1}[\sum_{i,j=1}^{r_1} A_{ij}^{(1)}x_ix_j+\sum_{i,j=r_1+1}^{n} B_{ij}^{(1)}x_ix_j], \tag{4}$$

where

$$B_{ij}^{(1)}\equiv\begin{vmatrix} A_{11}^{(1)}\ldots.A_{1r_1}^{(1)} & A_{1j}^{(1)} \\ \ldots\ldots\ldots\ldots \\ A_{r_11}^{(1)}\ldots.A_{r_1r_1}^{(1)} & A_{r_1j}^{(1)} \\ A_{i1}^{(1)}\ldots.A_{ir_1}^{(1)} & A_{ij}^{(1)} \end{vmatrix} M_{r_1}^{-1}$$

is congruent to an integer modulo P^λ and is divisible by P. If now all the $P^{e_1}B_{ij}^{(1)}$ are zero modulo P^λ, we have q_n replaced by

$$P^{e_1}\sum_{i,j=1}^{r_1} A_{ij}^{(1)}x_ix_j.$$

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* Using $P^{e_1}B_{jm}\equiv 0 \pmod{P^\lambda}$ in place of $B_{jm}\equiv 0 \pmod P$.

In the contrary case let e_2 denote the highest power of P that divides all the $P^{e_1}B_{ij}^{(1)}$ $(e_1<e_2<\lambda)$. Then we have (4) expressed in the form

$$P^{e_1}\sum_{i,\,j=1}^{r_1}A_{ij}^{(1)}x_ix_j+P^{e_2}\sum_{i,\,j=r_1+1}^{n}A_{ij}^{(2)}x_ix_j, \tag{5}$$

where at least one of the $A_{ij}^{(2)}$ is prime to P. The question now arises as to whether a different choice of M_{r_1} might not lead to a set of $B_{ij}^{(1)}$ that would give a different e_2. To answer this we note that, since M_{r_1} is prime to P and hence is congruent to a power ρ^m of a primitive root ρ of P^λ, we can transform (5) into

$$P^{e_1}[x_1^2+\ldots.+x_{r_1-1}^2+\rho^m x_{r_1}^2]+P^{e_2}\sum_{i,\,j=r_1+1}^{n}A_{ij}^{(2)}x_ix_j.$$

By applying a transformation of the type

$$x_{r_1}=\rho^{-k}x'_{r_1},\quad x_n=\rho^k x'_n,$$

we obtain one of the forms

$$\left.\begin{aligned}&P^{e_1}[x_1^2+\ldots.+x_{r_1}^2]+P^{e_2}q\quad (q\text{ free of }x_1,\ldots.,x_{r_1}),\\&P^{e_1}[x_1^2+\ldots.+x_{r_1-1}^2+\rho x_{r_1}^2]+P^{e_2}q_\rho\quad (q_\rho\text{ free of }x_1,\ldots.,x_{r_1}).\end{aligned}\right\} \tag{6}$$

If we had chosen M'_{r_1} instead of M_{r_1}, we would have obtained from (2) one of

$$\left.\begin{aligned}&P^{e_1}[y_1^2+\ldots.+y_{r_1}^2]+P^{e'_2}q'\quad (q'\text{ free of }y_1,\ldots.,y_{r_1}),\\&P^{e_1}[y_1^2+\ldots.+y_{r_1-1}^2+\rho y_{r_1}^2]+P^{e'_2}q'_\rho\quad (q'_\rho\text{ free of }y_1,\ldots.,y_{r_1}).\end{aligned}\right\} \tag{7}$$

Since* $\qquad x_1^2+\ldots.+x_{r_1}^2$ and $y_1^2+\ldots.+y_{r_1-1}^2+\rho y_{r_1}^2$

are not equivalent modulo P, it is evident that if we obtain (6_1) by the use of M_{r_1} we must obtain (7_1) by the use of M'_{r_1}. Similarly, if we obtain (6_2) by the use M_{r_1} we must obtain (7_2) by the use of M'_{r_1}. For definiteness, suppose that we obtain (6_1) and (7_1) and that $e_2<e'_2$. Then the forms

$$P^{e_1}(x_1^2+\ldots.+x_{r_1}^2)+P^{e_2}q\ \text{ and }\ P^{e_1}(y_1^2+\ldots.+y_{r_1}^2)$$

are transformable into each other modulo $P^{e'_2}$, whenever the forms

$$x_1^2+\ldots.+x_{r_1}^2+P^kq\ \text{ and }\ y_1^2+\ldots.+y_{r_1}^2$$

are transformable into each other modulo P^l, where $l=e'_2-e_1$ and $k=e_2-e_1$. Let the second of these be transformed into the first by

$$T:\ y_i\equiv\sum_{j=1}^{n}c_{ij}x_j\pmod{P^l}.$$

Employing partial derivatives with respect to x_i, we have

$$S:\left\{\begin{aligned}x_1&\equiv c_{11}y_1+\ldots.+c_{r_11}y_{r_1},\\&\ldots\ldots\ldots\ldots\ldots\ldots\\x_{r_1}&\equiv c_{1r_1}y_1+\ldots.+c_{r_1r_1}y_{r_1}\end{aligned}\right\}\pmod{P^l}.$$

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* L. E. Dickson, The Madison Colloquium Lectures, 1913, p. 8.

The inverse of S gives y_i $(i=1, \ldots, r_1)$ congruent to linear homogeneous functions of $x_1, \ldots, x_{r_1}$ modulo P^l. Hence, if (6_1) and (7_1) are equivalent modulo P^λ, $P^k q \equiv 0 \pmod{P^l}$ for all integral values of $x_{r_1+1}, \ldots, x_n$. Since at least one of the coefficients in q is prime to P, this implies that $k \geqq l$, *i. e.*, that $e_2 \geqq e_2'$. Similarly, e_2' can not be less than e_2. Therefore, the choice of the principal minor of $|A_{ij}^{(1)}|$ that is prime to P is immaterial so far as the value of e_2 is concerned. Consider now the two forms

$$Q = x_1^2 + \ldots + x_{r_1}^2 + P^l q, \quad Q' = y_1^2 + \ldots + y_{r_1}^2 + P^l q'.$$

Let Q and Q' be equivalent modulo $P^{\lambda - e_1}$ and let

$$T: \; y_i \equiv \sum_{j=1}^{n} c_{ij} x_j \pmod{P^{\lambda - e_1}} \qquad (i=1, \ldots, n)$$

transform Q' into Q modulo $P^{\lambda - e_1}$, where $|c_{ij}| \not\equiv 0 \pmod{P}$. Then we have

$$S: \left\{ \begin{aligned} \frac{\partial Q}{\partial x_1} &= 2x_1 \equiv 2c_{11}y_1 + \ldots + 2c_{r_1 1}y_{r_1} + P^l \frac{\partial q'}{\partial x_1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{\partial Q}{\partial x_{r_1}} &= 2x_{r_1} \equiv 2c_{1 r_1}y_1 + \ldots + 2c_{r_1 r_1}y_{r_1} + P^l \frac{\partial q'}{\partial x_{r_1}} \end{aligned} \right\} \pmod{P^{\lambda - e_1}},$$

which shows that $|c_{ij}|$ $(i, j=1, \ldots, r_1)$ is prime to P. The inverse of S gives $y_1, \ldots, y_{r_1}$ expressed as linear homogeneous functions of $x_1, \ldots, x_{r_1}$, $y_{r_1+1}, \ldots, y_n$. Moreover, the coefficients of $y_{r_1+1}, \ldots, y_n$ in these functions are divisible by P^l. Hence T is expressible in the form

$$y_i \equiv \sum_{j=1}^{r_1} c_{ij}x_j + P^l \sum_{j=r_1+1}^{n} c_{ij}' y_j \pmod{P^{\lambda - e_1}} \qquad (i=1, \ldots, r_1),$$

$$y_i \equiv \sum_{j=1}^{n} c_{ij}x_j \pmod{P^{\lambda - e_1}} \qquad (i=r_1+1, \ldots, n).$$

Subjecting Q' to this transformation, we have

$$\phi(x_1, \ldots, x_{r_1}) + P^l \theta(x_1, \ldots, x_n).$$

Since, by hypothesis, T transforms Q' into Q modulo $P^{\lambda - e_1}$, the terms in θ involving $x_1, \ldots, x_{r_1}$ must vanish modulo $P^{\lambda - e_1}$, and we have

$$\phi \equiv x_1^2 + \ldots + x_{r_1}^2 \pmod{P^{\lambda - e_1}}.$$

From this we see that the equivalence of (6_1) and (7_1) modulo P^λ implies the equivalence of $P^{e_2}q$ and $P^{e_2}q'$. Similarly, for the equivalence of (6_2) and (7_2). This shows that in the method of reduction of a quadratic form to the canonical form (8) or (9) modulo P^λ, it is immaterial whether we effect it by employing principal minors or by a seriatim process as was done by Jordan in a similar case.* This seriatim process is given in the argument leading up to (3).

* *Journal de Mathématiques*, Ser. 2, Vol. XVII (1872), pp. 368–375.

Repetitions of the arguments leading to (5) and (6) show that there exists a linear homogeneous transformation of determinant unity modulo P^λ under which, when $|P^{-e_1}a_{ij}|$ is divisible by P, the form (1) becomes

$$P^{e_1}[x_1^2+\ldots.+x_{r_1-1}^2+\rho^{m_1}x_{r_1}^2]+P^{e_2}[x_{r_1+1}^2+\ldots.+x_{r_1+r_2-1}^2+\rho^{m_2}x_{r_1+r_2}]$$
$$+\ldots.+P^{e_k}[x_{r_1+r_2+\ldots.+r_{k-1}+1}^2+\ldots.+x_{r_1+\ldots.+r_{k-1}}^2+\rho^{m_k}x_{r_1+\ldots.+r_k}^2] \quad (8)$$

if $r_1+r_2+\ldots.+r_k<n$, or

$$P^{e_1}[x_1^2+\ldots.+x_{r_1-1}^2+\rho^{m_1}x_{r_1}^2]+P^{e_2}[x_{r_1+1}^2+\ldots.+x_{r_1+r_2-1}^2+\rho^{m_2}x_{r_1+r_2}^2]$$
$$+\ldots.+P^{e_k}[x_{r_1+\ldots.+r_{k-1}+1}^2+\ldots.+x_{n-1}^2+P^{-\sum_{i=1}^{k}r_ie_i}\rho^{-\sum_{i=1}^{k-1}m_i}Dx_n^2] \quad (9)$$

if $r_1+r_2+\ldots.+r_k=n$, where $m_i=0$ or 1 [$i=1, \ldots., k$ in (8); $i=1, \ldots., k-1$ in (9)], $D=|a_{ij}|$ and r_s is the rank modulo P of the determinant

$$|A_{ij}^{(s)}| \ (i, j=r_1+r_2+\ldots.+r_{s-1}+1, \ldots., n).$$

$A_{ij}^{(s)}=P^{-e_s}B_{ij}^{(s-1)}$, where the $B_{ij}^{(s-1)}$ are found from the $A_{ij}^{(s-1)}$ in the same way that the $B_{ij}^{(1)}$ were found from $A_{ij}^{(1)}$ in (4). Therefore, when not all the a_{ij} are zero modulo P^λ, we have the canonical forms (3), (8), (9), and from the work above it is evident that two forms are equivalent modulo P^λ, if and only if they have the same canonical form. Since (3) is a special case of (9) with $k=1$, *i. e.*, $r_1=n$, we need consider only the canonical forms (8), (9) and the vanishing form where $a_{ij}\equiv 0 \pmod{P^\lambda}$ $(i, j=1, \ldots., n)$. Moreover, since two forms are equivalent modulo P^λ if and only if they have the same canonical form, we have the classes C_0 and $Ce_im_iR_k$, defined by a canonical form of type (8) with $n>R_k=r_1+\ldots.+r_k$; Ce_im_jnD, defined by a canonical form of type (9) with $n=R_k$, for $i=1, \ldots., k$; $j=1, \ldots., k-1$; $e_i<\lambda$.

2. *Elementary Exponents and Ranks.*—Consider the symmetric determinant $D=|a_{ij}|$ $(i, j=1, \ldots., n)$, where the elements are integers. If P^{α_i} is the highest power of P that divides all the i-rowed minors of D, then α_i is unchanged when D is pre-multiplied or post-multiplied by an n-th order determinant whose elements are integers and whose value is unity.* Frobenius called the P^{e_i} $(e_i=\alpha_i-\alpha_{i-1}, e_1=\alpha_1)$ the elementary invariants of D with respect to P. We shall call the e_i the elementary exponents of D with respect to P, or simply the elementary exponents of D. Note for later use that† e_i is a monotonically increasing function of i. Let P^{e_1} be the highest power of P that divides all the a_{ij} and consider the case when $e_1<\lambda$. Let the rank modulo P of

* Stephen Smith, *Phil. Trans.*, Vol. CLI, § 12, p. 311. Bachmann's *Zahlentheorie*, Vol. IV. Frobenius, *Berlin Sitzungsberichte* (1894), Pt. I, p. 31. These properties hold if D is not symmetric, but our interest lies in the theorem as given.

† Frobenius, *loc. cit.* Also seen from this section.

the determinant $|A_{ij}|=|P^{-e_1}a_{ij}|$ be r_1, $0<r_1<n$. Then there exists* a principal minor M_{r_1} of $|A_{ij}|$ of order r_1 that is prime to P. By multiplying D by properly chosen unit determinants we may assume that

$$|A_{ij}|=M_{r_1}\not\equiv 0 \pmod P \qquad (i, j=1, \ldots, r_1).$$

Consider the unit n-rowed determinant $d^{(1)}=|d^{(1)}_{st}|$ in which each element is zero except

$$d^{(1)}_{st}=1 \text{ when } s=t, \quad d^{(1)}_{r_1+1,\,t}\equiv -\frac{A_{1,\,r_1+1}\overline{A}_{1t}}{M_{r_1}} \pmod{P^\lambda} \qquad (t=1, \ldots, r_1),$$

where $\overline{A}_{ij}$ denotes the cofactor of A_{ij} in M_{r_1}. If we pre-multiply D by $d^{(1)}$ we have, upon denoting the resulting elements by $b^{(1)}_{ij}$,

$$b^{(1)}_{1,\,r_1+1}\equiv a_{1,\,r_1+1}-\frac{a_{1,\,r_1+1}}{M_{r_1}}\sum_{i=1}^{r_1}A_{1i}\overline{A}_{1i}\equiv 0 \pmod{P^\lambda},$$

$$b^{(1)}_{k,\,r_1+1}\equiv a_{k,\,r_1+1}-\frac{a_{1,\,r_1+1}}{M_{r_1}}\sum_{i=1}^{r_1}A_{ki}\overline{A}_{1i} \pmod{P^\lambda} \qquad (k\neq 1).$$

Hence

$$b^{(1)}_{k,\,r_1+1}\equiv a_{k,\,r_1+1} \pmod{P^\lambda} \qquad (1<k\leqq r_1),$$

$$b^{(1)}_{k,\,r_1+1}\equiv\frac{P^{e_1}}{M_{r_1}}[A_{k,\,r_1+1}M_{r_1}-A_{1,\,r_1+1}M^{(k)}_{r_1 1}] \qquad (k>r_1),$$

where $M^{(k)}_{r_1 1}$ denotes the determinant formed from M_{r_1} by replacing its first row or column by the corresponding elements of the k-th row or column of $|A_{ij}|$ $(i, j=1, \ldots, n)$. All other elements of D are unchanged. If we now pre-multiply this resulting determinant by $d^{(2)}=|d^{(2)}_{st}|$, where $s, t=1, \ldots, n$, and every element is zero except

$$d^{(2)}_{st}=1 \text{ when } s=t; \quad d^{(2)}_{r_1+1,\,t}\equiv -\frac{A_{2,\,r_1+1}\overline{A}_{2t}}{M_{r_1}} \qquad (t=1, \ldots, r_1),$$

we have

$$b^{(2)}_{1,\,r_1+1}\equiv b^{(2)}_{2,\,r_1+1}\equiv 0 \pmod{P^\lambda}, \quad b^{(2)}_{k,\,r_1+1}\equiv a_{k,\,r_1+1} \qquad (k=3, 4, \ldots, r_1),$$

$$b^{(2)}_{k,\,r_1+1}\equiv\frac{P^{e_1}}{M_{r_1}}[A_{k,\,r_1+1}M_{r_1}-A_{1,\,r_1+1}M^{(k)}_{r_1 1}-A_{2,\,r_1+1}M^{(k)}_{r_1 2}] \pmod P \qquad (k>r_1).$$

All other elements are again unchanged. Continuing this process, we ultimately obtain from D a determinant having its first r_1 elements in the (r_1+1)-th row zero modulo P^λ and the k-th element of this row $(k>r_1)$ is

$$P^{e_1}B^{(1)}_{k,\,r_1+1}\equiv\begin{vmatrix} A_{11} & \ldots . A_{1r_1} & A_{1k} \\ \ldots & \ldots\ldots & \ldots \\ A_{r_1 1} & \ldots . A_{r_1 r_1} & A_{r_1 k} \\ A_{r_1+1,\,1} & \ldots . A_{r_1+1,\,r_1} & A_{r_1+1,\,k} \end{vmatrix} P^{e_1}M^{-1}_{r_1} \pmod{P^\lambda}.$$

* L. E. Dickson, *Annals of Mathematics*, Ser. 2, Vol. XV (1913), pp. 27, 28.

$B^{(1)}_{k,r_1+1}$ is divisible by $P(r_1<k\leqq n)$, since r_1 is the rank modulo P of the determinant $|A_{ij}|$ $(i, j=1, \ldots, n)$. All the elements of D in any other row are unchanged. Operating similarly upon the (r_1+2)-th, ..., n-th row of this determinant and then post-multiplying by a properly chosen unit determinant, we ultimately obtain from D

$$D^{(1)}\equiv\begin{vmatrix} A & O \\ O & B \end{vmatrix} \pmod{P^\lambda}, \quad A=\begin{pmatrix} a_{11} \ldots . a_{1r_1} \\ \ldots\ldots\ldots \\ a_{r_11} \ldots . a_{r_1r_1} \end{pmatrix}, \quad B=\begin{pmatrix} P^{e_1}B^{(1)}_{r_1+1r_1+1} \ldots . P^{e_1}B^{(1)}_{r_1+1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ P^{e_1}B^{(1)}_{nr_1+1} \ldots . P^{e_1}B^{(1)}_{nn} \end{pmatrix},$$

where O is a matrix all of whose elements are zero.

Here $B^{(1)}_{ij}$ $(i, j=r_1+1, \ldots, n)$ is divisible by P. To determine the highest power of P that divides all the $P^{e_1}B^{(1)}_{ij}$, let $P^{e'_2}$ denote the highest power of P that divides all the (r_1+1)-rowed minors of D. Then $P^{e'_2}$ is the highest power of P that divides all the (r_1+1)-rowed minors of $D^{(1)}$. Then evidently $P^{e'_2-e_1r_1}$ is the highest power of P that divides all the $P^{e_1}B^{(1)}_{ij}$ of $D^{(1)}$. Let $e_2=e'_2-r_1e_1$. Then e_2 is the (r_1+1)-th elementary exponent of D. We shall call r_1 the first elementary rank of D with respect to P. Consider the case when $e_2<\lambda$. Let r_2 be the rank modulo P of the determinant

$$|P^{-(e_2-e_1)}B^{(1)}_{ij}| \qquad (i, j=r_1+1, \ldots . n).$$

Since there is a principal minor of this determinant of order r_2 that is prime to P, we may operate on this determinant as we did on D above. Moreover, considering this determinant as it stands in $D^{(1)}$, we see that this operation would leave the first r_1 rows and columns of $D^{(1)}$ unaltered. Hence we would have

$$D^{(2)}\equiv\begin{vmatrix} A & O & O \\ O & B^{(1)} & O \\ O & O & C \end{vmatrix} \pmod{P^\lambda},$$

$$B^{(1)}=\begin{pmatrix} P^{e_1}B^{(1)}_{r_1+1, r_1+1} \ldots . P^{e_1}B^{(1)}_{r_1+1, r_1+r_2} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ P^{e_1}B^{(1)}_{r_1+r_2, r_1+1} \ldots . P^{e_1}B^{(1)}_{r_1+r_2, r_1+r_2} \end{pmatrix}, \quad C=\begin{pmatrix} P^{e_2}B^{(2)}_{r_1+r_2+1, r_1+r_2+1} \ldots . P^{e_2}B^{(2)}_{r_1+r_2+1, n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ P^{e_2}B^{(2)}_{n, r_1+r_2+1} \ldots\ldots . P^{e_2}B^{(2)}_{n, n} \end{pmatrix}.$$

Let $P^{e'_3}$ be the highest power of P that divides all the (r_1+r_2+1)-rowed minors of D. Then $e'_3-r_1e_1-r_2e_2=e_3$ is the highest power of P that divides all the $P^{e_2}B^{(2)}_{ij}$ of $D^{(2)}$. Evidently we may continue this process until all the new elements are zero modulo P^λ, or until $\Sigma r_i=n$. At any stage, say the t-th, e_t and r_t are determined by the preceding. In fact

$$e_t=e'_t-\sum_{i=1}^{t-1} r_ie_i,$$

where e'_t is the highest power of P that divides all the $(r_1+r_2+\ldots .+r_{t-1}+1)$-rowed minors of D, and r_t is an integer such that $E=r_1e_1+\ldots .+r_te_t$ is the

exponent of the highest power of P that divides all the $(r_1+r_2+\ldots.+r_t)$-rowed minors of D while all the $(r_1+\ldots.+r_t+1)$-rowed minors are divisible by a higher power of P than P^{E+e_t}. We shall refer to these r_i as the elementary ranks of D with respect to P.

Definition. The determinant D shall be said to be of rank R modulo P^λ if the R-th elementary exponent of D with respect to P is less than λ, while the $(R+1)$-th elementary exponent of D is $\geqq\lambda$.

From this definition it follows that the rank of D modulo P^λ is an invariant under multiplication of D by unit determinants and that under such multiplication D becomes

$$D^{(k)}\equiv\begin{vmatrix} A & O & O\ldots.O & O & O\ldots. \\ O & B^{(1)} & O\ldots.O & O & O\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ O & O & O\ldots.O & B^{(k)}O\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{vmatrix} \pmod{P^\lambda},$$

$$B^{(k)}=\begin{pmatrix} P^{e_k}B^{(k)}_{R_k+1,\,R_k+1}\cdot\ldots.P^{e_k}B^{(k)}_{R_k+1,\,R_{k+1}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P^{e_k}B^{(k)}_{R_{k+1},\,R_k+1}\ldots.P^{e_k}B^{(k)}_{R_{k+1},\,R_{k+1}} \end{pmatrix},$$

where $R_i=r_1+\ldots.+r_i$, and R_k is the rank of D modulo P^λ. The $e_1, e_2, \ldots., e_k$ are the distinct elementary exponents of D that are less than λ arranged in the order of increasing magnitude. From these considerations it follows that if the rank modulo P^λ of a symmetric determinant D is $R>0$, there exists a principal minor M_R of D of order R whose rank modulo P^λ is R, and such that the highest power of P that divides all the R-rowed minors of D is exactly the power of P that is contained in M_R, or, more precisely, whose elementary exponents are those of D that are less than λ and whose elementary ranks are precisely those of D. In fact, the M_{R_k} formed by taking the first R_k rows and columns of $D^{(k)}$ is congruent modulo P^λ to such a principal minor of D of order R_k. From the form of $D^{(k)}$ also follows the fact that the elementary exponent is a monotonically increasing function of the order of the minors.

3. *Criteria for the Classes $Ce_im_iR_k$ and Ce_im_jnD.*—It follows from §§ 1 and 2 that the determinant D of a quadratic form, the e_i, r_i of D and the rank $R_k=r_1+\ldots.+r_k$ of D modulo P^λ are invariants. However, these invariants do not completely characterize the classes. We seek criteria for the determination of the m_i of (8) and (9) § 1. Consider the modular form q_n given by (1) and let not all the a_{ij} be zero modulo P^λ. Under transformations of the type used in § 1, q_n becomes

$$\sum_{s=1}^{k}P^{e_s}\sum_{i,\,j=R_{s-1}+1}^{R_s}A^{(s)}_{ij}x_ix_j,\quad R_s=\sum_{i=1}^{s}r_i, \tag{10}$$

where $e_1<e_2<\ldots.<e_k$ are the distinct elementary exponents of D with respect to P that are less than λ, and the r_i are the elementary ranks of D with respect to P. The determinants $|A_{ij}^{(s)}|$ are prime to P, and R_k is the rank modulo P^λ of D.

Consider first the case $R_k<n$. Let $E_i=r_1e_1+\ldots.+r_ie_i$ and denote the minor $|P^{e_1}A_{ij}^{(1)}|$ of the determinant of (10) by M_{R_1}. Then q_n belongs to the class $Ce_im_iR_k$ with $m_1=0$ or 1, according as $(P^{-E_1}M_{R_1})^{\frac{\mu}{2}}\equiv 1$ or -1 (mod P^λ), where $\mu=\phi(P^\lambda)=P^{\lambda-1}(P-1)$. By §§ 1, 2, M_{R_1} is congruent modulo P^λ to an R_1-rowed principal minor of D such that $P^{-E_1}M_{R_1}$ is prime to P. Let now M_{R_1} denote any such principal minor of D. Then, since q_n has one and but one of the canonical forms, we conclude that q_n belongs to the class $Ce_im_iR_k$ with $m_1=0$ or 1 according as $(P^{-E_1}M_{R_1})^{\frac{\mu}{2}}\equiv 1$ or -1 (mod P^λ). Consider next the minor

$$M_{R_2}\equiv\begin{vmatrix} P^{e_1}A_{11}^{(1)} & \ldots. P^{e_1}A_{1,R_1}^{(1)} & 0 \ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots \\ P^{e_1}A_{R_1,1}^{(1)} & \ldots. P^{e_1}A_{R_1R_1}^{(1)} & 0 \ldots\ldots\ldots\ldots 0 \\ 0 \ldots\ldots\ldots 0 \ldots\ldots & & P^{e_2}A_{R_1+1,\,R_1+1}^{(2)} \ldots. P^{e_2}A_{R_1+1,\,R_2}^{(2)} \\ \ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots\ldots\ldots \\ 0 \ldots\ldots\ldots 0 \ldots\ldots & & P^{e_2}A_{R_2,\,R_1+1}^{(2)} \ldots\ldots P^{e_2}A_{R_2R_2}^{(2)} \end{vmatrix}.$$

M_{R_2} is an R_2-rowed principal minor of the determinant of (10) and is congruent modulo P^λ to an R_2-rowed principal minor of D. Moreover, by §§ 1, 2, q_n belongs to the class $Ce_im_iR_k$ with $m_2=0$ or 1 according as

$$(|A_{ij}^{(2)}|)^{\frac{\mu}{2}}=\left(\frac{P^{-E_2}M_{R_2}}{|A_{ij}^{(1)}|}\right)^{\frac{\mu}{2}}\equiv(|A_{ij}^{(1)}|)^{\frac{\mu}{2}}(P^{-E_2}M_{R_2})^{\frac{\mu}{2}}\equiv 1 \text{ or } -1 \text{ (mod } P^\lambda).$$

Let now M_{R_2} denote any R_2-rowed principal minor of D having an R_1-rowed principal minor M_{R_1} such that $P^{-E_2}M_{R_2}$ and $P^{-E_1}M_{R_1}$ are prime to P. Then, in view of §§ 1, 2, and the fact that q_n has one and but one of the canonical forms, q_n belongs to the class $Ce_im_iR_k$ with $m_2=0$ or 1 according as

$$(P^{-E_1}M_{R_1})^{\frac{\mu}{2}}(P^{-E_2}M_{R_2})^{\frac{\mu}{2}}\equiv 1 \text{ or } -1 \text{ (mod } P^\lambda).$$

In general let M_{R_s} be any R_s-rowed principal minor of D having an R_{s-1}-rowed principal minor $M_{R_{s-1}}$, which in turn has an R_{s-2}-rowed principal minor $M_{R_{s-2}}$ and so on to M_{R_2} which has an R_1-rowed principal minor M_{R_1} such that $P^{-E_i}M_{R_i}$ $(i=1, \ldots., s)$ is prime to P. Then by a transformation of determinant unity which merely permutes the x_i of (1) we have the principal minors M_{R_1} occupying the first R_1 rows and columns of D, etc. Then from the form of $D^{(k)}$ of § 2 we have that q_n belongs to the class $Ce_im_iR_k$ with $m_s=0$ or 1 according as

$$\prod_{j=1}^{s}(P^{-E_j}M_{R_j})^{\frac{\mu}{2}}\equiv 1 \text{ or } -1 \text{ (mod } P^\lambda). \qquad (11)$$

For the case $R_k=n$, D together with (11) for $s=1, \ldots, k-1$ furnish criteria. At the first consideration one would probably be surprised that D may be zero modulo P^λ and still q_n belong to class Ce_im_jnD. In fact D may be divisible by a much higher power of P than P^λ and still q_n belong to class Ce_im_jnD, since it is not the power of P that divides the minors of D that determine the number of terms in the canonical form, but the elementary exponents of D. This phenomenon is what gave rise to the definition of the rank of D modulo P^λ of § 2. The rank of D modulo P^λ shall be called the rank of the form q_n modulo P^λ. Two forms q_n and q'_n of rank n are equivalent if and only if

$$e_i=e'_i,\ r_i=r'_i,\ m_j=m'_j \qquad (i=1, \ldots, k;\ j=1, \ldots, k-1)$$

and $P^{-E_k+e_k}D\equiv P^{-E'_k+e'_k}D'$ (mod P^λ) as is seen by §§ 1, 2.

4. *Invariantive Criteria for the Classes.*—Consider the modular quadratic form (1), and let $e_1<e_2<\ldots<e_h$ be the distinct elementary exponents of D with respect to P. Let $\overline{D}$, $\bar{e}_i$, $\bar{r}_i$ and $\overline{m}_i$ be a consistent set of values that define a class $C\bar{e}_i\overline{m}_i\bar{R}_k$ or $C\bar{e}_i\overline{m}_jn\overline{D}$ according as the rank $\bar{R}_k$ modulo P^λ of $\overline{D}$ is less than or equal to n. For abbreviation set

$$\begin{aligned}
&\mathfrak{R}_i=\sum_{j=1}^{i-1} r_j+\bar{r}_i, && \mathfrak{E}_i=\sum_{j=1}^{i-1} r_je_j+\bar{r}_ie_i && (i=1, \ldots, k),\\
&\bar{R}_i=\sum_{j=1}^{i} \bar{r}_j, && \bar{E}_i=\sum_{j=1}^{i} \bar{r}_j\bar{e}_j && (i=1, \ldots, k),\\
&R'_i=\sum_{j=1}^{i-1} r_j+r'_i, && E'_i=\sum_{j=1}^{i-1} r_je_j+r'_ie_i && (i=1, \ldots, k;\ \bar{r}_i<r'_i\leqq n-R_{i-1}),\\
&R_i=\sum_{j=1}^{i} r_j, && E_i=\sum_{j=1}^{i} r_je_j && (i=1, \ldots, h).
\end{aligned}$$

Thus R_h and $\bar{R}_k$ are the ranks modulo P^λ of D and $\overline{D}$, respectively.

Consider the function *

$$\begin{aligned}
A_{\bar{R}_1}(D)\equiv{}&(P^{e_1-\bar{r}_1})^\mu\{(P^{-\mathfrak{E}_1}M^{(1)}_{\mathfrak{R}_1})^{\frac{\mu}{2}}+(P^{-\mathfrak{E}_1}M^{(2)}_{\mathfrak{R}_1})^{\frac{\mu}{2}}[1-(P^{-\mathfrak{E}_1}M^{(1)}_{\mathfrak{R}_1})^\mu]\\
&+\ldots+(P^{-\mathfrak{E}_1}M^{(r)}_{\mathfrak{R}_1})^{\frac{\mu}{2}}\prod_{j=1}^{r-1}[1-(P^{-\mathfrak{E}_1}M^{(j)}_{\mathfrak{R}_1})^\mu]\}\prod_i[1-(P^{-E'_1}M^{(i)}_{R'_1})^\mu] \pmod{P^\lambda}, \quad (12)
\end{aligned}$$

where $M^{(j)}_{\mathfrak{R}_1}$ ranges over the principal minors of D of order $\mathfrak{R}_1=\bar{R}_1$, and $M^{(j)}_{R'_1}$ ranges over the principal minors of D of order $R'_1>\mathfrak{R}_1$. Since P^{e_1} divides all the a_{ij} of D, it is evident that there is no term of $A_{\bar{R}_1}(D)$ that has as factor P with a negative exponent. $A_{\bar{R}_1}(D)$ is zero modulo P^λ, unless $e_1=\bar{e}_1$ and unless at least one of the $P^{-\mathfrak{E}_1}M_{\mathfrak{R}_1}$ is prime to P (since $P\geqq 3$). This implies that $A_{\bar{R}_1}(D)\equiv 0 \pmod{P^\lambda}$ unless the rank modulo P of the determinant $|P^{-e_1}a_{ij}|$ is $\bar{r}_1$. In fact, if $r_1<\bar{r}_1$, none of the $P^{-\mathfrak{E}_1}M_{\mathfrak{R}_1}$ are prime to P, while if $r_1>\bar{r}_1$, at least one

* In case $k=1$ and $\lambda>1$, $A_{\bar{R}_1}(D)$ is to have also the final factor of (15).

of the $P^{-E'_1}M_{R'_1}$ is prime to P and the final factor is zero by Fermat's theorem. Thus the value of $A_{\overline{R}_1}(D)$ is zero modulo P^λ, unless q_n belongs to a class $Ce_im_iR_k$ or Ce_im_jnD with $e_1=\overline{e}_1$ and $r_1=\overline{r}_1$, and by (11), § 3, its value modulo P^λ is 1 or -1 according as $m_1=0$ or 1.

Consider next the function

$$A_{\overline{R}_2}(D)\equiv(P^{e_2-\overline{r}_2})^\mu\{A_{\overline{R}_1}(M^{(1)}_{\mathfrak{R}_2})\,(P^{-\mathfrak{E}_2}M^{(1)}_{\mathfrak{R}_2})^{\frac{\mu}{2}}+A_{\overline{R}_1}(M^{(2)}_{\mathfrak{R}_2})\,(P^{-\mathfrak{E}_2}M^{(2)}_{\mathfrak{R}_2})^{\frac{\mu}{2}}[1-(P^{-\mathfrak{E}_2}M^{(1)}_{\mathfrak{R}_2})^\mu]$$
$$+\ldots.+A_{\overline{R}_1}(M^{(s)}_{\mathfrak{R}_2})(P^{-\mathfrak{E}_2}M^{(s)}_{\mathfrak{R}_2})^{\frac{\mu}{2}}\prod_{j=1}^{s-1}[1-(P^{-\mathfrak{E}_2}M^{(j)}_{\mathfrak{R}_2})^\mu]\}\prod_i[1-P^{-E'_2}M^{(i)}_{R'_2})^\mu]\ (\mathrm{mod}\,P^\lambda),\quad(13)$$

where $M^{(j)}_{\mathfrak{R}_2}$ ranges over the principal minors of D of order $\mathfrak{R}_2$, and $M^{(i)}_{R'_2}$ ranges over the principal minors of D of orders $R'_2>\mathfrak{R}_2$. That the principal minors are divisible by the power of P indicated in (13) follows from the fact that the elementary exponents of D increase monotonically with the order of the minors to which they correspond. In fact, by hypothesis, every minor of order R_1 of D is divisible by $P^{r_1e_1}$ while every minor of order R_1+1 is divisible by $P^{r_1e_1+e_2}$. Let ε be the (R_1+2)-th elementary exponent of D. Then every minor of D of order R_1+2 is divisible by $P^{r_1e_1+e_2+\varepsilon}$. From the fact that $\varepsilon\geqq e_2$ we see that every minor of D of order R_1+2 is divisible by $P^{r_1e_1+2e_2}$. Continuing this we have that every minor of D of order $R_1+\overline{r}_2=\mathfrak{R}_2$ is divisible by $P^{r_1e+r_2e_2}=P^{\mathfrak{E}_2}$. If none of the $P^{-\mathfrak{E}_2}M_{\mathfrak{R}_2}$ are prime to P, $A_{\overline{R}_2}(D)\equiv0\ (\mathrm{mod}\,P^\lambda)$. Its value is also zero modulo P^λ unless $r_1=\overline{r}_1$, $e_1=\overline{e}_1$ and $e_2=\overline{e}_2$ from the fact that each term carries an $A_{\overline{R}_1}$ function as factor and from the first factor. Thus $A_{\overline{R}_2}(D)$ is zero modulo P^λ unless at least one of the $M^{(j)}_{\mathfrak{R}_2}$ is such that $P^{-\mathfrak{E}_2}M_{\mathfrak{R}_2}$ is prime to P and at the same time contains a principal minor $M_{\mathfrak{R}_1}$ such that $P^{-\mathfrak{E}_1}M_{\mathfrak{R}_1}$ is prime to P. Let all these conditions be satisfied and take this $P^{-\mathfrak{E}_1}M_{\mathfrak{R}_1}=P^{-\overline{E}_1}M_{\overline{R}_1}$ as the M_{R_1} of §§ 1 and 2. Then we have that q_n is equivalent modulo P^λ to a form whose determinant is $D^{(1)}$ of § 2. If $r_2<\overline{r}_2$, all the $\mathfrak{R}_2$-rowed minors of $D^{(1)}$ are divisible by a higher power of P than $P^{\mathfrak{E}_2}$, and hence none of the $P^{-\mathfrak{E}_2}M_{\mathfrak{R}_2}$ of D are prime to P. Therefore, $r_2\geqq\overline{r}_2$. If $r_2>\overline{r}_2$, at least one of the R'_2-rowed principal minors of $D^{(1)}$ that contains $M_{\mathfrak{R}_1}$ as its leading R_1-rowed principal minor is such that $P^{-E'_2}M_{R'_2}$ is prime to P, and $A_{\overline{R}_2}(D)$ is zero modulo P^λ from its final factor by Fermat's Theorem. Hence $A_{\overline{R}_2}(D)\equiv0\ (\mathrm{mod}\,P^\lambda)$ unless q_n belongs to a class $Ce_im_iR_k$ or Ce_im_jnD with $e_1=\overline{e}_1$, $e_2=\overline{e}_2$, $r_1=\overline{r}_1$, $r_2=\overline{r}_2$, and by (11) its value is 1 or -1 modulo P^λ according as $m_2=0$ or 1. Similarly, we have that the function $(i=1,\ldots.,k-1)$

$$A_{\overline{R}_i}(D)\equiv(P^{e_i-\overline{r}_i})^\mu\{\prod_{j=1}^{i-1}A_{\overline{R}_j}(M^{(1)}_{\mathfrak{R}_i})\,(P^{-\mathfrak{E}_i}M^{(1)}_{\mathfrak{R}_i})^{\frac{\mu}{2}}$$
$$+\prod_{j=1}^{i-1}A_{\overline{R}_j}(M^{(2)}_{\mathfrak{R}_i})\,(P^{-\mathfrak{E}_k}M^{(2)}_{\mathfrak{R}_i})^{\frac{\mu}{2}}[1-(P^{-\mathfrak{E}_i}M^{(1)}_{\mathfrak{R}_i})^\mu]$$
$$+\ldots.+\prod_{j=1}^{i-1}A_{\overline{R}_j}(M^{(t)}_{\mathfrak{R}_i})\,(P^{-\mathfrak{E}_i}M^{(t)}_{\mathfrak{R}_i})^{\frac{\mu}{2}}\prod_{v=1}^{t-1}[1-(P^{-\mathfrak{E}_i}M^{(v)}_{\mathfrak{R}_i})^\mu]\}\prod_w[1-(P^{-E'_i}M^{(w)}_{R'_i})^\mu]$$
$$(\mathrm{mod}\,P^\lambda),\quad(14)$$

where $M^{(v)}_{\Re_i}$ ranges over the principal minors of D of order $\Re_i$, and $M^{(w)}_{R'_i}$ ranges over the principal minors of D of order $R'_i > \Re_i$, has the value zero unless q_n belongs to a class $Ce_i m_i R_k$ or to a class $Ce_i m_j n D$ with $e_j = \bar{e}_j$, $r_j = \bar{r}_j$, and that for such forms its value is 1 or -1 modulo P^λ according as $m_i = 0$ or 1. When $\bar{R}_k < n$, we have

$$\begin{aligned} A_{\bar{R}_k}(D) \equiv (P^{e_k - \bar{e}_k})^\mu \Big\{ \prod_{i=1}^{k-1} A_{\bar{R}_i}(M^{(1)}_{\Re_k}) \, (P^{-\mathfrak{E}_k} M^{(1)}_{\Re_k})^{\frac{\mu}{2}} \\ + \prod_{i=1}^{k-1} A_{\bar{R}_i}(M^{(2)}_{\Re_k}) \, (P^{-\mathfrak{E}_k} M^{(2)}_{\Re_k})^{\frac{\mu}{2}} [1 - (P^{-\mathfrak{E}_k} M^{(1)}_{\Re_k})^\mu] \\ + \ldots + \prod_{i=1}^{k-1} A_{\bar{R}_i}(M^{(t)}_{\Re_k}) \, (P^{-\mathfrak{E}_k} M^{(t)}_{\Re_k})^{\frac{\mu}{2}} \prod_{j=1}^{t-1} [1 - (P^{-\mathfrak{E}_k} M^{(j)}_{\Re_k})^\mu] \Big\} \\ \prod_v [1 - (P^{-E'_k} M^{(v)}_{R'_k})^\mu] \prod_w [1 - (P^{-(E_k + e_{k+1})} d^{(w)}_{R_k+1})^\mu] \pmod{P^\lambda}, \quad (15) \end{aligned}$$

where $M^{(j)}_{\Re_k}$ ranges over the principal minors of D of order $\Re_k$, $M^{(v)}_{R'_k}$ ranges over those of order $R'_k > \Re_k$, $d^{(w)}_{R_k+1}$ ranges over the minors of D of order $R_k + 1$, and e_{k+1} is the (R_k+1)-th elementary exponent of D. In case $e_{k+1} \geqq \lambda$, define $P^{-(E_k + e_{k+1})} d^{(w)}_{R_k+1}$ to be zero modulo P^λ. Then $A_{\bar{R}_k}(D)$ has the value zero unless $e_i = \bar{e}_i$, $r_i = \bar{r}_i$ $(i = 1, \ldots, k)$ and $e_{k+1} \geqq \lambda$, whence $R_k = R_k = \bar{R}_k$. Hence $A_{\bar{R}_k}(D) \equiv 0 \pmod{P^\lambda}$ unless q_n belongs to a class $Ce_i m_i R_k$ with $e_i = \bar{e}_i$ and $r_i = \bar{r}_i$ $(i = 1, \ldots, k)$, and for such forms its value is 1 or -1 according as $m_k = 0$ or 1. When $\bar{R}_k = n$ we have

$$\begin{aligned} A_n(D) \equiv (P^{e_k - \bar{e}_k})^\mu (P^{-E_k} D)^\mu \{1 - [(P^{-E_k + e_k} D - P^{-\bar{E}_k + \bar{e}_k} \bar{D}) P^{-q}]^\mu \Big\} \prod_{i=1}^{k-1} A_{\bar{R}_i}(D) \\ \pmod{P^\lambda}, \quad (16) \end{aligned}$$

where P^q is the highest power of P that divides $P^{-E_k + e_k} D - P^{-\bar{E}_k + \bar{e}_k} \bar{D}$. In case $q \geqq \lambda$, define $P^{-q}(P^{-E_k + e_k} D - P^{-\bar{E}_k + \bar{e}_k} \bar{D})$ to be zero modulo P^λ. Then the value of $A_n(D)$ is zero modulo P^λ unless $e_i = \bar{e}_i$, $r_i = \bar{r}_i$ $(i = 1, \ldots, k)$ and $P^{-E_k + e_k} D \equiv P^{-\bar{E}_k + \bar{e}_k} \bar{D} \pmod{P^\lambda}$. The function is also zero unless $P^{-E_k} D$ is prime to P, *i. e.*, unless the rank modulo P^λ of D is R_k. Hence $A_n(D)$ is zero modulo P^λ unless q_n belongs to a class $Ce_i m_j n D$ with $e_i = \bar{e}_i$, $r_i = \bar{r}_i$ $(i = 1, \ldots, k)$ and its value is 1 modulo P^λ for all such forms.

Finally, define $P^{-e_1} a_{ij}$ to be zero modulo P^λ when $e_1 \geqq \lambda$. Then

$$I \equiv \Pi [1 - (P^{-e_1} a_{ij})^\mu] \pmod{P^\lambda} \qquad (i, j = 1, \ldots, n; \; i \leqq j). \quad (17)$$

has the value zero modulo P^λ unless q_n belongs to the class C_0 and the value 1 for such a form. The functions $A_{\bar{R}_i}(D)$, $A_n(D)$ and I are modular invariants that completely characterize the classes of modular quadratic forms modulo P^λ.

5. *Characteristic Modular Invariants.*—An invariant ϕ that has the value 1 for all forms of one class, and the value zero for forms of any other

class is called a characteristic invariant. In case such a ϕ is a modular invariant it is called a characteristic modular invariant. For example, I is a characteristic invariant for the class C_0.

Let $\bar{e}_i, \bar{r}_i, \bar{m}_i$ $(i=1, \ldots, k)$ be a set of consistent values that define the class $C\bar{e}_i\bar{m}_i\bar{R}_k$. Among the integers $1, \ldots, k$ a certain number s_1 of them will correspond to m's that are zero. The remaining s_2 of them will correspond to m's that are unity. Call the first set σ_1, and the second, σ_2, and consider the function

$$A_{\bar{R}_k\sigma_1\sigma_2} \equiv \frac{1}{2^{s_1+s_2}} \prod_i [(A_{\bar{R}_i}(D))^2 + A_{\bar{R}_i}(D)] \prod_j [(A_{\bar{R}_j}(D))^2 - A_{\bar{R}_j}(D)] \pmod{P^\lambda}, \quad (18)$$

where i ranges over σ_1, and j ranges over σ_2. This function has the value 1* for all forms of class $C\bar{e}_i\bar{m}_i\bar{R}_k$ and the value zero for all other forms. Hence $A_{\bar{R}_k\sigma_1\sigma_2}$ is a characteristic modular invariant of q_n.

Next let $\bar{e}_i, \bar{r}_i$ $(i=1, \ldots, k)$, $\bar{m}_j$ $(j=1, \ldots, k-1)$ and $\bar{D}$ be a consistent set of values of these quantities that define the class $C\bar{e}_i\bar{m}_jn\bar{D}$; thus $\bar{R}_k=n$. Separate the integers $1, \ldots, k-1$ into sets σ_1 and σ_2 composed of s_1 and s_2 elements, respectively, such that those of σ_1 correspond to the m's that are zero, while those of σ_2 correspond to the m's that are unity. Then the function

$$A_{n\sigma_1\sigma_2} \equiv \frac{1}{2^{s_1+s_2}} \prod_i [(A_{\bar{R}_i}(D))^2 + A_{\bar{R}_i}(D)] \prod_j [(A_{\bar{R}_j}(D))^2 - A_{\bar{R}_j}(D)] A_n(D) \pmod{P}, \quad (19)$$

where i ranges over σ_1 and j ranges over σ_2, has the value 1 for all forms of class $C\bar{e}_i\bar{m}_jn\bar{D}$ and the value zero for all other forms.

Hence the characteristic invariants $A_{\bar{R}_k\sigma_1\sigma_2}$, $A_{n\sigma_1\sigma_2}$, and I completely characterize the classes of modular quadratic forms modulo P^λ.

For the case $\lambda=1$, $A_{\bar{R}_0}(D)$ and $A_{\bar{R}_k\sigma_1\sigma_2}$ become, respectively, the invariants A_r and $I_{r,\pm 1}$ given by Professor Dickson in his Madison Colloquium lectures, pp. 11, 13. The function $A_{n\sigma_1\sigma_2}$ is for this case a characteristic invariant of his class C_{nD}, while I in this case is replaced by the simple form $I=\Pi(1-a_{ij}^{\mu})$, as given *loc. cit.*, p. 11. These are the only ones that enter in this case, since the only set of integers e_i, m_i, r_i are $0, r_1, m_1$. Thus, for this case the invariants given above are all polynomial modular invariants. This may also happen for the case $\lambda>1$, but would obviously be a very special case.

6. *Number of Linearly Independent Modular Invariants of q_n.*—For convenience denote the classes of modular quadratic forms modulo P^λ by $C_0, C_1, C_2, \ldots$ and the corresponding characteristic invariants by

* In case either σ_1 or σ_2 is an empty set, the corresponding product factors of (18) is defined to be unity. Similarly in (19).

$I_0, I_1, I_2, \cdots\cdots$. Since any modular invariant I of q_n takes certain values $v_0, v_1, v_2, \ldots.$ for the respective classes $C_0, C_1, C_2, \ldots.$ we have

$$I \equiv v_0 I_0 + v_1 I_1 + v_2 I_2 + \ldots. \pmod{P^\lambda}.$$

Hence, any modular invariant is congruent to a linear homogeneous function of the characteristic invariants. Moreover, the number of linearly independent modular invariants of the quadratic form q_n modulo P^λ equals the number of classes.

II. Polynomial Modular Invariants of a Binary Quadratic Form.

7. *Invariants for a Prime Power Modulus P^λ.*—Consider the form

$$f = a_0 x^2 + a_1 xy + a_2 y^2, \tag{1}$$

where the a_i are integers taken modulo P^λ. The transformations *

$$\begin{aligned} R&: x = x' + Ty', \quad y = y', \\ S&: x = x', \quad y = ky', \quad (k \text{ prime to } P) \\ Q&: x = y', \quad y = -x', \end{aligned}$$

generate the group of all linear homogeneous transformations whose determinant is prime to P and coefficients are integers modulo P^λ. Transformations R, S, Q give the respective replacements

$$a_0' = a_0, \quad a_1' = a_1 + 2a_0 T, \quad a_2' = a_2 + a_1 T + a_0 T^2. \tag{2}$$

$$a_0' = a_0, \quad a_1' = k a_1, \quad a_2' = k^2 a_2. \tag{3}$$

$$a_0' = a_2, \quad a_1' = -a_1, \quad a_2' = a_0. \tag{4}$$

Theorem 1. *The form f has the absolute invariant modulo P^λ*

$$H = \prod_{i=0}^{2} (a_i^\mu - 1) \qquad \mu = \phi(P^\lambda) = P^{\lambda-1}(P-1).$$

Evidently replacements (3) and (4) leave H unaltered. It remains to show that H is unaltered by replacement (2). This is evident if a_0 is prime to P, since then $a_0^\mu - 1 \equiv 0 \pmod{P^\lambda}$. Hence let a_0 be divisible by P. Then a_1' is of the form $a_1 + Pq$ and, by induction on λ,

$$(a_1 + Pq)^{P^{\lambda-1}} \equiv a_1^{P^{\lambda-1}} \pmod{P^\lambda},$$

whence $a_1'^\mu \equiv a_1^\mu$. Hence $H' \equiv A(a_2'^\mu - 1)$, where

$$A = (a_0^\mu - 1)(a_1^\mu - 1).$$

Thus $H' \equiv H$ if either a_0 or a_1 is prime to P. In the contrary case, $a_2 = a_2 + Pq$ and $a_2'^\mu \equiv a_2^\mu$ as before, whence $H' \equiv H$ in all cases.

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* Cf. C. Jordan, "Traité des Substitutions," p. 93.

The same method leads to

THEOREM 2. *The form f has the absolute invariants modulo* 2^λ

$$I=a_1^\mu=a_1^{2^{\lambda-1}}, \quad Q=a_0^\mu a_1^\mu a_2^\mu.$$

8. *Fundamental System of Invariants Modulo* 4.

THEOREM 3. *Necessary and sufficient conditions that*

$$\alpha+\beta T+\gamma T^2+\delta T^3\equiv 0 \pmod 4$$

for all integral values of T are

$$\alpha\equiv 0, \quad 2\beta\equiv 2\gamma\equiv\beta+\gamma+\delta\equiv 0 \pmod 4.$$

The general invariants of § 7 become

$$H=\prod_{i=0}^{2}(a_i^2-1), \quad I=a_1^2, \quad Q=a_0^2a_1^2a_2^2.$$

A useful combination of these invariants is

$$S=P-Q-I+1=a_0^2+a_2^2-a_0^2a_2^2-a_0^2a_1^2-a_1^2a_2^2.$$

THEOREM 4. *The form f has the absolute invariants* *

$$\begin{aligned}K&\equiv a_1+a_1^3+2a_0a_2,\\ J&\equiv 2a_0+2a_2+a_0a_1+a_0a_1^2+a_1a_2+a_1a_2^3+a_0a_1a_2+a_0a_1^3a_2,\\ E&\equiv a_0a_2+a_0^2a_2^2+a_0a_2(a_1+a_1^2+a_1^3)\\ &\equiv a_0^3a_2+a_0a_2^2+a_0^2a_2^2-a_0^3a_2^2+a_0a_1a_2^2+a_0^3a_1^2a_2+a_0^3a_1^3a_2^2 \pmod 4.\end{aligned}$$

The following identical congruences are of use in simplifying the discussion given below for the general invariant.

$$U\equiv(a_1+a_1^2)(a_2+a_2^2)\equiv 0, \quad V\equiv(a_0+a_0^2)(a_1+a_1^2)\equiv 0, \quad L\equiv(a_0+a_0^2)(a_2+a_2^2)\equiv 0,$$

$$Q_1=3Q\equiv a_0^3a_1^2a_2^2+a_0^3a_1^3a_2^2+a_0^2a_1^3a_2^2, \quad Q_2=3Q\equiv a_0^2a_1^2a_2+a_0^3a_1^2a_2+a_0^3a_1^2a_2^2,$$

$$M\equiv a_0a_1a_2^2+a_0^3a_1a_2^2+a_0a_1^2a_2+a_0^3a_1^2a_2\equiv 4Q\equiv 0,$$

$$N\equiv a_0a_1a_2^2+a_0^3a_1a_2^2+a_0a_1^3a_2+a_0^3a_1^3a_2\equiv 4Q\equiv 0.$$

In view of the identical congruence $x^4\equiv x^2 \pmod 4$, we may assume that the exponent of each a_i in an invariant of f does not exceed 3. Hence, we may denote any invariant ϕ by

$$\phi\equiv\sum_{i,j=0}^{3}A_{ij}a_1^ia_2^j, \quad A_{ij}=\alpha_{ij}+\beta_{ij}a_0+\gamma_{ij}a_0^2+\delta_{ij}a_0^3, \tag{5}$$

where α_{ij}, etc., are constants. The difference

$$\phi-(A_{11}+A_{13}a_2+A_{31}a_1+A_{33}a_1a_2)U$$

has

$$A_{11}\equiv A_{13}\equiv A_{31}\equiv A_{33}\equiv 0.$$

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* *K, J* and *E* were discovered by carrying through for special cases the discussion made below for finding the general invariant; they were verified to be invariant with respect to the generators *R, S, Q*.

Hence, we may assume that in the initial ϕ these quantities are zero modulo 4. Similarly, by considering in turn

$$\phi-[\beta_{01}+(\gamma_{01}-\beta_{01})a_0]L-(\delta_{01}-\gamma_{01}+\beta_{01})E,\quad \phi-\alpha_{10}K-[\beta_{10}+(\gamma_{10}-\beta_{10})a_0]V,$$
$$\phi-[\beta_{30}+(\gamma_{30}-\beta_{30})a_0]a_1V,\qquad \phi-[\beta_{03}+(\gamma_{03}-\beta_{03})a_0]a_2L,$$

we may assume that

$$A_{01}\equiv\alpha_{01},\quad A_{10}\equiv\delta_{10}a_0^3,\quad A_{30}\equiv\alpha_{30}+\delta_{30}a_0^3,\quad A_{03}\equiv\alpha_{03}+\delta_{03}a_0^3.$$

Then by employing in turn

$$\phi-\beta_{21}M-\gamma_{21}Q_2,\quad \phi-\beta_{32}N-\gamma_{32}Q_1,\quad \phi-\alpha_{12}J-\gamma_{12}Q_2,\quad \phi-\alpha_{20}I-\gamma_{22}Q+\alpha_{22}S,$$

we may take

$$A_{21}\equiv\alpha_{21}+\delta_{21}a_0^3,\quad A_{32}\equiv\alpha_{32}+\delta_{32}a_0^3,\quad A_{12}\equiv\beta_{12}a_0+\delta_{12}a_0^3,\quad \alpha_{20}=\alpha_{22}=\gamma_{22}=0.$$

At no step have any of the quantities previously taken to be zero been disturbed.

Subjecting the simplified ϕ to replacement (2) we get

$$\phi'\equiv\phi+\beta T+\gamma T^2+\delta T^3,$$

where

$$\begin{aligned}\beta\equiv{}&2A_{10}a_0+2A_{12}a_0+2A_{02}a_0a_2+2A_{12}a_0a_2+(A_{01}+2A_{02}a_0+2A_{22}a_0+2A_{30}a_0\\&+2A_{32}a_0+2A_{02}a_2+2A_{03}a_0a_2+2A_{12}a_2+2A_{12}a_0a_2+2A_{22}a_2+2A_{22}a_0a_2\\&+2A_{23}a_0a_2+2A_{32}a_2+3A_{03}a_2^2)a_1+(A_{21}+3A_{23}a_2^2)a_1^3,\\ \gamma\equiv{}&A_{01}a_0+A_{02}a_0^2+A_{03}a_0^3+3A_{03}a_0a_2^2+3A_{03}a_0^2a_2+A_{12}a_0^2a_1+(A_{02}+3A_{03}a_2+A_{21}a_0\\&+3A_{03}a_0+A_{22}+A_{22}a_0^2+3A_{23}a_0a_2^2+3A_{23}a_2+3A_{23}a_0^2a_2+3A_{23}a_0+A_{23}a_0^3)a_1^2\\&+(A_{12}+A_{32}+A_{32}a_0^2)a_1^3,\\ \delta\equiv{}&3A_{03}a_0^2a_1+(A_{03}+A_{23}+3A_{23}a_0^2)a_1^3.\end{aligned}$$

By Theorem 3, necessary and sufficient conditions that $\phi'-\phi\equiv0$ for all integral values of T are $2\beta\equiv2\gamma\equiv\beta+\gamma+\delta\equiv0$. From the form of A_{10}, A_{12} and A_{22} we see that

$$2(A_{12}+A_{12}a_0)\equiv0,\quad 2(A_{22}+A_{22}a_0)\equiv0,\quad 2(A_{10}+A_{10}a_0)\equiv0. \tag{6}$$

From $2\beta\equiv2(A_{01}+A_{21})a_1+2(A_{03}+A_{23})a_1a_2$, we have

$$2(A_{01}+A_{21})\equiv0,\quad 2(A_{03}+A_{23})\equiv0. \tag{7}$$

Next,

$$2\gamma\equiv2(A_{01}+A_{02}+A_{03})a_0+2[A_{02}+A_{22}+A_{12}+A_{32}+a_0(A_{12}+A_{21}+A_{03}+A_{22}+A_{32})]a_1.$$

Hence, in view of (7), and (6),

$$2(A_{01}+A_{02}+A_{03})a_0\equiv0,\quad 2(A_{02}+A_{32})(1+a_0)\equiv0. \tag{8}$$

Making use of the above relations, $\beta+\gamma+\delta\equiv 0$ gives

$$\begin{aligned}&2A_{10}a_0+2A_{12}a_0+A_{01}a_0+A_{02}a_0^2+A_{03}a_0^3+a_2(2A_{02}a_0+2A_{12}a_0+3A_{03}a_0^2)+3A_{03}a_0a_2^2\\&+[A_{01}+2A_{02}a_0+2A_{22}a_0+2A_{30}a_0+2A_{32}a_0+A_{12}a_0^2+3A_{03}a_0^2+a_2(2A_{02}+2A_{32})\\&+3A_{03}a_2^2]a_1+[A_{02}+A_{22}+A_{21}a_0+3A_{03}a_0+A_{22}a_0^2+3A_{23}a_0+A_{23}a_0^3\\&\qquad+a_2(3A_{03}+3A_{23}+3A_{23}a_0^2)+3A_{23}a_0a_2^2]a_1^2\\&+[A_{21}+A_{12}+A_{32}+A_{03}+A_{23}+3A_{23}a_0^2+A_{32}a_0^2+3A_{23}a_2^2]a_1^3\equiv 0.\end{aligned}$$

Denote this congruence by $p+qa_1+ra_1^2+sa_1^3\equiv 0$. Hence $p\equiv 2q\equiv 2r\equiv q+r+s\equiv 0$. From $p\equiv 0$ we have

$$\left.\begin{aligned}&2A_{10}a_0+2A_{12}a_0+A_{01}a_0+A_{02}a_0^2+A_{03}a_0^3\equiv 0,\quad 2A_{03}a_0\equiv 0,\\&2A_{02}a_0+2A_{12}a_0+A_{03}a_0^2+A_{03}a_0\equiv 0.\end{aligned}\right\}\tag{9}$$

From $2q\equiv 0$ we have

$$2(A_{01}+A_{12}a_0)\equiv 0,\quad 2A_{03}\equiv 0.\tag{10}$$

By (7) we have $2A_{23}\equiv 0$. Hence $A_{23}a_1^2a_2^3\equiv A_{23}a_1^2a_2^2$. Hence we may take $\alpha_{23}\equiv 0$ by replacing ϕ by $\phi+\alpha_{23}S$, and $\gamma_{23}\equiv 0$ by subtracting $\gamma_{23}Q$. $A_{23}a_1^2a_2^2$ is now in a form that can be combined with $A_{22}a_1^2a_2^2$. Hence we may assume $A_{23}\equiv 0$. Similarly, by combining $A_{03}a_2^3$ with $A_{02}a_2^2$ we may take $A_{03}\equiv 0$. It is readily verified that the above reductions do not disturb the quantities that we have previously taken to be zero. A like remark applies to the reductions that follow.

Since $A_{01}\equiv\alpha_{01}$, we have, from (10), that $2A_{01}\equiv 0$. Take $A_{01}=0$ by combining $A_{01}a_2\equiv A_{01}a_2^2$ with $A_{02}a_2^2$. From (7) we now have that $2A_{21}\equiv 0$. Hence $A_{21}a_1^2a_2\equiv A_{21}a_1^2a_2^2$ and we may take $\alpha_{21}\equiv 0$ by considering $\phi+\alpha_{21}S$. We now have $A_{21}a_1^2a_2\equiv\delta_{21}a_0^3a_1^2a_2^2$ and may combine it with $A_{22}a_1^2a_2^2$ and take $A_{21}\equiv 0$. From (10) and (6) we have $2A_{12}\equiv 0$. Hence, combining $A_{12}a_1a_2^2\equiv A_{12}a_1^2a_2^2$ with $A_{22}a_1^2a_2^2$, we may take $A_{12}\equiv 0$. The above conditions now become

$$\left.\begin{aligned}&2A_{02}+2A_{32}(1+a_0)\equiv 0,\quad 2A_{10}a_0+A_{02}a_0^2\equiv 0,\\&A_{01}\equiv A_{03}\equiv A_{12}\equiv A_{21}\equiv A_{23}\equiv 0,\quad 2(A_{22}+A_{22}a_0)\equiv 0.\end{aligned}\right\}\tag{11}$$

From $2r\equiv 0$ we have $2(A_{02}+A_{22}+A_{22}a_0)\equiv 0$, whence $2A_{02}\equiv 0$. From $q+r+s\equiv 0$ we have

$$2A_{22}a_0+2A_{30}a_0+2A_{32}a_0+A_{02}+A_{22}+A_{22}a_0^2+A_{32}+A_{32}a_0^2+2A_{32}a_2\equiv 0.\tag{12}$$

Hence $2A_{32}\equiv 0$. We may take $\alpha_{32}\equiv 0$ by considering $\phi+\alpha_{32}S$. By combining $A_{32}a_1^3a_2^2\equiv\delta_{32}a_0^3a_1^2a_2^2$ with $A_{22}a_1^2a_2^2$ we have $A_{32}\equiv 0$. By the final condition (11), (12) now becomes

$$A_{02}+A_{22}-A_{22}a_0^2+2A_{30}a_0\equiv 0.\tag{13}$$

From (11) we have

$$2\alpha_{02}\equiv 2(\beta_{02}+\delta_{02})\equiv 2\gamma_{02}\equiv 2\delta_{10}+\alpha_{02}+\beta_{02}+\gamma_{02}+\delta_{02}\equiv 0.\tag{14}$$

From (13) we obtain the additional conditions

$$\alpha_{02}\equiv 2(\beta_{02}+\beta_{22})\equiv 2(\alpha_{30}+\delta_{30})+\beta_{02}+\gamma_{02}+\delta_{02}\equiv 0. \tag{15}$$

From (14) and (15) we have

$$\alpha_{02}\equiv 2\gamma_{02}\equiv 2\delta_{10}+\beta_{02}+\gamma_{02}+\delta_{02}\equiv 2(\beta_{02}+\beta_{22})\equiv 2(\delta_{10}+\alpha_{30}+\delta_{30})\equiv 0. \tag{16}$$

We now have

$$\begin{aligned}\phi=\alpha_{00}&+\beta_{00}a_0+\gamma_{00}a_0^2+\delta_{00}a_0^3+\beta_{02}a_0a_2^2+\gamma_{02}a_0^2a_2^2+\delta_{02}a_0^3a_2^2+\delta_{10}a_0^3a_1\\&+\beta_{20}a_0a_1^2+\gamma_{20}a_0^2a_1^2+\delta_{20}a_0^3a_1^2+\beta_{22}a_0a_1^2a_2^2+\delta_{22}a_0^3a_1^2a_2^2+\alpha_{30}a_1^3+\delta_{30}a_0^3a_1^3,\end{aligned}$$

where the Greek letter constants satisfy relations (16). Let ϕ become ϕ' under the replacement (4). Then $\phi-\phi'=p+qa_2+ra_2^2+sa_2^3$, where

$$\begin{aligned}p&=\beta_{00}a_0+\gamma_{00}a_0^2+\delta_{00}a_0^3+\delta_{10}a_0^3a_1+2\alpha_{30}a_1^3+\delta_{30}a_0^3a_1^3+\beta_{20}a_0a_1^2+\gamma_{20}a_0^2a_1^2+\delta_{20}a_0^3a_1^2,\\ q&=-\beta_{00}-\beta_{02}a_0^2-\beta_{20}a_1^2-\beta_{22}a_0^2a_1^2,\\ r&=\beta_{02}a_0+\delta_{02}a_0^3-\gamma_{00}+\beta_{22}a_0a_1^2+\delta_{22}a_0^3a_1^2-\gamma_{20}a_1^2,\\ s&=-\delta_{00}-\delta_{02}a_0^2+\delta_{10}a_1-\delta_{20}a_1^2-\delta_{22}a_0^2a_1^2+\delta_{30}a_1^3.\end{aligned}$$

Then $p\equiv 0$ gives $\beta_{00}\equiv\gamma_{00}\equiv\delta_{00}\equiv 0$ and

$$2\delta_{10}\equiv 2(\beta_{20}+\gamma_{20}+\delta_{20})\equiv 2\delta_{30}\equiv 0,\quad 2\alpha_{30}+\beta_{20}a_0+\gamma_{20}a_0^2+(\delta_{10}+\delta_{20}+\delta_{30})a_0^3\equiv 0,$$

whence

$$2\alpha_{30}\equiv 2\beta_{20}\equiv 2\delta_{20}\equiv 2\gamma_{20}=0.$$

Hence $A_{20}\equiv 2N_{20}a_0^3$. Take $A_{20}\equiv 0$ by combining $2N_{20}a_0^3a_1^2\equiv 2N_{20}a_0^3a_1$ with $\delta_{10}a_0^3a_1$. Also take $\alpha_{30}\equiv 0$ by considering $\phi-\alpha_{30}I$. The remaining condition from $p\equiv 0$ is $\delta_{10}\equiv\delta_{30}$. Hence ϕ has the term $\delta_{10}(a_0^3a_1+a_0^3a_1^3)\equiv 0$. Thus ϕ is congruent to an expression with $\delta_{10}\equiv\delta_{30}\equiv 0$. From $2r\equiv 0$ we have $2(\beta_{22}+\delta_{22})\equiv 0$. From $q+r+s\equiv 0$ we have

$$\beta_{02}a_0-(\beta_{02}+\delta_{02})a_0^2+\delta_{02}a_0^3+[\beta_{22}a_0-(\beta_{22}+\delta_{22})a_0^2+\delta_{22}a_0^3]a_1^2\equiv 0,$$

whence

$$2\beta_{02}\equiv 2\delta_{02}\equiv 2\beta_{22}\equiv 2\delta_{22}\equiv 0.$$

From the first two of these, together with $2\gamma_{02}\equiv 0$ and $\beta_{02}+\gamma_{02}+\delta_{02}\equiv 0$, from (16), (since $2\delta_{10}\equiv 0$), we have $A_{02}\equiv 0$. From $2\beta_{22}\equiv 2\delta_{22}\equiv 0$ we have

$$\beta_{22}a_0a_1^2a_2^2+\delta_{22}a_0^3a_1^2a_2^2\equiv(\beta_{22}+\delta_{22})a_0^2a_1^2a_2^2\equiv(\beta_{22}+\delta_{22})Q,$$

and hence $\phi\equiv\alpha_{00}+(\beta_{22}+\delta_{22})Q$. From this we have

Theorem 5. *Any modular invariant of a binary quadratic form whose coefficients are integers modulo* 4 *is a linear function of the invariants* I, H, Q, J, K, E.

The binary form is congruent to a multiple of a square of a linear form if and only if $I\equiv K\equiv E\equiv 0 \pmod 4$.